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ABSTRACT

This is a self-contained manual for use by teachers in preparation for classroom presentations. One of the goals of the report is to show how games and puzzles can provide effective means for developing mathematical understanding and skills. The authors indicate that this type of activity is well adapted for discovery teaching techniques. The report is organized into two main parts. The first part contains experimental units that were tested in the classroom. The topics in this part include: (1) nim-type games; (2) polyominoes; (3) symmetry; (4) a counting machine; (5) finding the greatest common divisor; (6) linear function games; and (7) games with addition tables. The second part consists of the report of a project to compile a list of games and puzzles appropriate for use in the mathematics classroom. Twenty-seven papers contain (in addition to the above list) activities such as: (1) magic squares; (2) Fibonacci problems; (3) geometric puzzles; (4) numerical oddities; and (5) powers and primes. (MP)

SCHOOL MATHEMATICS STUDY GROUP

Studies in Mathematics

VOLUME XVIII

Puzzle Problems and Games Project

Final Report

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Final Report

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[illegible]

INTRODUCTION

[illegible] Project spon- [illegible]

[illegible] number of mathe- [illegible] provide [illegible] understanding and skills. [illegible] educational value. [illegible] literature on mathe- [illegible] would be appropriate for [illegible] conference [illegible] mathematicians with [illegible] mathematical recreations [illegible] of the literature, [illegible] developing new ideas [illegible] conference, consisting [illegible] working papers, [illegible] Appendix I of this report.

[illegible] educational potential [illegible] classroom use. The idea must [illegible] the students' [illegible] game or puzzle problem [illegible] training session for [illegible] particular game or puzzle will be [illegible] project was to pre- [illegible] games and puzzles which were self- [illegible] in order to be [illegible].

In order to [illegible] of the project, four teams, each consisting of [illegible] mathematicians [illegible] educator, were organized. Each team was [illegible] appropriate written material from one or two [illegible] first phase of the project, [illegible] writing [illegible] winter and spring of 19[illegible]. In [illegible] 19[illegible] [illegible] week tryout and evalu- ation session. [illegible] volunteered for [illegible] experiment. [illegible] topics developed [illegible]

[illegible]

[illegible] potential [illegible] In particular, [illegible] tradition must be [illegible]

[illegible] the [illegible]

[illegible]

[illegible] and [illegible] and to [illegible] more [illegible]

[illegible] pantomimes and [illegible] before present- [illegible] that they only understand [illegible] pantomimes [illegible] young [illegible] these con- [illegible]

[illegible]

[illegible] student-players to [illegible] principles. One teacher [illegible] taking place in the one-hour [illegible] months of the [illegible] this fifth grade class who [illegible] suddenly [illegible] skill while participating in [illegible]

[illegible] When they are adapted to [illegible] the game is well [illegible] of the play,

APPENDIX A

[illegible] PROBLEMS AND GAMES PROJECT

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[illegible] in the tryout [illegible].

APPENDIX [illegible]

[illegible]

NIM-TYPE GAMES

(Prepared by [illegible] Ellsworth and [illegible] Hill)

INTRODUCTION

Nim-type games are played by two players and may be described as follows: A starting sequence of numbers is given; the players play alternately, each play consisting of a modification of the sequence of numbers within limits prescribed by the rules of the game. The first player to change the sequence into one of a given set of target configurations wins the game.

Perhaps the simplest Nim-type game is that in which the starting sequence consists of the single number zero. Each play consists of adding one of the numbers 1, [illegible] to the number previously obtained. The first player to reach [illegible] given target number, for example [illegible], wins.

A slightly more complicated game of this type is the game of "21". Again the starting sequence consists of the single number zero. Each play consists of adding one of the numbers 1, [illegible] to the number already obtained. The target number in this game is the number 21.

In the classical game of Nim, the starting sequence consists of the three numbers [illegible]. Each play consists of replacing any one of the numbers in the sequence by a smaller whole number. The first player to obtain the sequence [illegible].

[illegible] as follows: The starting sequence [illegible] numbers [illegible] 1, [illegible] 2. Each play consists [illegible] of the numbers of the sequence. The [illegible] the number 1 wins.

[illegible] need not be played with numbers. [illegible] players are provided with a [illegible]. The first player selects a rod and [illegible]. The second player then selects a rod and [illegible] rod already played in [illegible]. The players [illegible] play in this manner. The player [illegible].

The [illegible] wide variety of [illegible].

Experience has shown that games of this type are capable of stimulating the interest of [illegible] of nearly every age [illegible]. However, we are not principally interested in the entertainment value of the games, but rather in ways in which they can contribute to mathematical skills and understanding.

The [illegible] Nim-type games [illegible] in elementary addition and subtraction. As such, they are probably applicable only to the early elementary grades. Increasing the target number and altering the selection set, while complicating the arithmetic, [illegible] appreciably to the interest of the game. Indeed, by greatly increasing the number of possible alternatives, it may simply make the game more confusing to the players. However, the games can always be [illegible] the level of arithmetical skill of the players.

[illegible] the most significant mathematical contribution of Nim-type games [illegible] of problem solving. [illegible] the development of a strategy [illegible] examples of problems in which elementary but non-[illegible] in [illegible] solutions. The [illegible] which occur in many [illegible] progress toward solution [illegible] -- [illegible]. In fact, [illegible] the player can win every [illegible].

[illegible] problem [illegible] of [illegible] the student [illegible]. On the other hand, the [illegible] development of a [illegible] for solving the [illegible].

[illegible] problems [illegible] little [illegible] this aspect [illegible] winning [illegible].

[illegible]

[illegible] could be con- [illegible]

rules, [illegible] judgment as to when to [illegible] variation or new [illegible], and perhaps to discuss [illegible] strategy with the players.

It may be advisable, at some point, to play [illegible] team games, either with the [illegible] divided into two [illegible] teams and players taking turns [illegible], or [illegible] small groups [illegible] consulting on their plays [illegible] spectators. The [illegible] advantage of the former is that [illegible] the disadvantage is that one cannot [illegible] every play. The small-group [illegible] compromise, [illegible] discuss and agree upon their plays [illegible].

An example of [illegible] chance to participate is the following:

[illegible] is divided into two teams. Each team consists of pairs of players who work together. [illegible] of matches between pairs, [illegible] play one match and [illegible] point [illegible]. At the end of all matches [illegible].

For example, [illegible] with a selection set of {1, 2}. The first pair [illegible] the first pair on team B. The [illegible]. In the meantime, all other players [illegible] from standing closely to the [illegible]. [illegible] the second pair [illegible] until all pairs have played [illegible] not talk to their partner [illegible] with other players [illegible].

[illegible] scored by the team [illegible]

[illegible]

[illegible] simple, but non-trivial [illegible] following: Let the [illegible] and let the selection set [illegible] of the numbers [illegible] adds it to the number selected by the first player. The first player then again selects 1 or 2 and adds it to the sum obtained by the second player. They continue in this manner until one of the players makes the sum equal to 7. This player is the winner.

It is important for the students to have some experience playing the game and making decisions since this is the only way they will discover that there is indeed a problem to be solved in winning the game and thus be motivated to try to find a solution. The first prerequisite in understanding a game is actual experience in playing the game. This is important for the teacher since without this experience it will be difficult to appreciate the pupils' choice of a method of play and their approach to an understanding of the game.

Since this is a fairly simple game, some pupils may be able to discover a winning strategy after playing relatively few games. These pupils should be warned not to give away the winning strategy to the other pupils. On the other hand, [illegible] the winning strategy by a few of the students may [illegible] other pupils [illegible] the game.

[illegible]

If many of the students fail to find quickly a winning strategy for themselves, the teacher is presented with an excellent opportunity to illustrate one of the basic principles of problem solving: _If a problem does not yield [illegible], [illegible] a simpler problem which has many of the features of the original problem._ In [illegible], the teacher may suggest that the pupils try the game in which the sum [illegible] is smaller, say 4. For this game, the pupils should quickly discover that choosing [illegible] is a poor first move since the [illegible] and win. Hence, the [illegible] choice of the second player, the first player can always make the sum equal to 4. Having found the winning strategy [illegible] the pupils should now be able to [illegible] the target number 7. In fact, since they now [illegible] possible for the first player to make the sum equal to [illegible], it is [illegible] of what number [illegible] the sum equal to 7 and thus win. [illegible] winning first move is to choose the number 1.

[illegible]

important that they continue until they clearly realize that the winning play consists of [illegible] 1 [illegible] the first play and then making [illegible] sum of 4 and finally [illegible].

It [illegible] also worthwhile for the students [illegible] discover that if the first player [illegible] makes [illegible] play, the second player can always go on to win. When the students have pretty well mastered this simple game, it is probably the appropriate time to propose the problem: How does the winning play change if the target number is changed to [illegible]? This question should test whether they really understand the principle of the winning strategy. If they do, then they should be able to decide quickly what changes should be made in the winning strategy. The students may also try the target numbers 9, 10, and 11 to further reinforce their understanding. Finally, the value of a general description of the winning strategy [illegible] by setting a very high target number, say [illegible]. Here the pupils may discover the important principle that the [illegible] by dividing the target number by [illegible].

GAMES WITH SELECTION SET [illegible] 1, ,

After the solution to the game with the selection set {1,..} is clearly understood, the next natural step is to change the game in another direction, namely, change the selection set. [illegible] for developing general techniques [illegible] in all of mathematics. This may be done by [illegible] the target number to 14 and changing the selection set to {1, , } [illegible] directly to the game [illegible], the pupils should begin by playing [illegible] develop a winning strategy [illegible] the results which they [illegible] to [illegible] them to the new [illegible] to make conjectures and to test them [illegible] noted that this approach [illegible] in mathematics.

In the [illegible] for the pupils to discover [illegible] the number [illegible] and that the number [illegible] by a multiple [illegible] discover that [illegible] the first player [illegible].

For example, by trying the number 1 at the beginning, he may lose but in the play he may notice that [illegible] would have been a better move because he noticed a pattern in the results. In the next play he may try this to see if his conjecture about the pattern works. Such a conclusion may then apply in principle to other variations. This discovery of a strategy may be entirely individual or it may be discovered by a team of players.

There is probably [illegible] in asking a student who has discovered a winning strategy [illegible] as to why it works. In this way he will be making the first step in formulating a mathematical proof. The teacher's judgment must determine when this is appropriate and to whom the strategy is revealed. For example, it will be of more benefit to other students to continue playing and trying to discover a strategy for themselves. The child who discovers the winning strategy in play may first discuss his argument with the teacher and then leave the game to the others. On the other hand, a team effort in finding, conjecturing and testing may be valuable experience. The main objective is that the discovery of a strategy of play is a product of each child's effort and not just what someone else tells him.

OTHER MODIFICATIONS

From this point on, there are a wide variety of modifications of the game which lead to new problems in determining the winning strategy.

SUBTRACTION GAME

The game can be changed to one of subtraction, namely by setting the beginning number to be a positive whole number and, in each play, one of the numbers in the collection is subtracted. The winning number in this case is zero.

OTHER [illegible]

Instead of [illegible] a consecutive set of integers beginning with 1, [illegible] consecutive set beginning with another number, for example, [illegible]. If the beginning number is zero and the [illegible] the first choice is [illegible] and the winning numbers for the first player [illegible] multiple of [illegible]. It should be noted that this game may [illegible] if the [illegible] strategy is now followed. For example, if the [illegible] it is impossible to get the sum [illegible]1. Also note that if [illegible] then it is impossible for the first player [illegible] by choosing [illegible] or [illegible] in the first play. [illegible] players will discover these facts for themselves. A very [illegible] at this point might [illegible] to describe a general [illegible] and appropriate target numbers.

[illegible] entirely new [illegible].

TWO-[illegible]

[illegible] consist of [illegible] of the numbers [illegible] in the sequence. Hence, [illegible] and the [illegible] added. The [illegible] The analysis of this [illegible] win. Hence, it is better [illegible] first play is to add 1 to the [illegible] {1,1}. The other winning [illegible] {[illegible]}, {[illegible]}, [illegible] {[illegible]}. It should be noted that the [illegible] of the corresponding [illegible].

[illegible]

[illegible], [illegible] more numbers, the [illegible], in the classical game of Nim in which [illegible] in each play, any one number [illegible] it does not become negative, the [illegible] follows: [illegible] each number in the [illegible] is a winning combination if [illegible] number of [illegible] in the three [illegible].

[illegible] to determine and [illegible]. This necessitates [illegible] to analyze, and [illegible] difficult enough to challenge [illegible] enough to make solutions possible. [illegible] a new variation, different [illegible] and interest and to provide [illegible] the principle to a new game. He must also decide how best to obtain the dual values of independent discovery of strategy and the systematization of the conjectures and verification accomplished by a convincing argument as to why it works.

POLYOMINOES

(Prepared by R. P. Dilworth and J. Hill)

BACKGROUND MATERIAL FOR THE TEACHER

When geometry is applied in the physical sciences the relevant feature of a physical object is its shape. If an object is moved to a different position in space, its geometric properties are not altered. Hence, in the study of geometry, it is important to know whether or not a given configuration can be made to match another configuration by means of such a motion. It may even happen that a given configuration can be made to match itself under certain motions. In this case, the configuration will have certain symmetries. These symmetries, in turn, may determine important physical properties of objects having this shape.

In this unit a class of simple configurations will be introduced and the ways in which these configurations can be matched and fit together will be considered. These activities will provide a natural introduction to some of the basic ideas of geometry.

The configurations to be studied are those configurations in the plane which can be constructed by joining congruent squares along edges. Such configurations are called <u>polyominoes</u>. If n squares are used in constructing the configuration, it will be convenient to refer to the polyomino as an <u>n-square</u>.

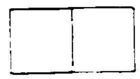
The simplest configuration consists of a single square, i.e., the <u>monomino</u> or <u>one-square</u>. With regard to the motions of these configurations, there are two observations which should be made.

1. If two congruent squares are given in a plane, it is always possible to take one of them, turn it, and then move it across the plane in such a way that it matches exactly the second square.

2. If a square is given a quarter turn (clockwise) about its center, then it exactly matches the square in its first position. Likewise, a half turn, three quarter turn, or full turn about the center also brings the square into a matching position.

It may also be noted that a quarter turn counterclockwise produces the same result as a three quarter turn clockwise.

The next simplest configuration is the <u>domino</u> or <u>two-square</u> which is obtained by joining the edges of two congruent squares.



The matching of dominoes is quite similar to the matching of single squares.

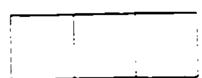
1. Given two dominoes, it is always possible to take one of them, turn it, and then move it across the plane in such a way that it matches the second domino.

A domino no longer matches itself if it is turned a quarter turn about its center. The appropriate observation for dominoes is the following:

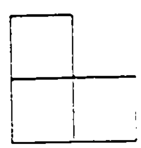
- If a domino is given a half turn or full turn about its center, it exactly matches the domino in its first position.

The plane configuration constructed by joining the edges of three squares is called a <u>tromino</u> or <u>three-square</u>.

One such configuration is the following:

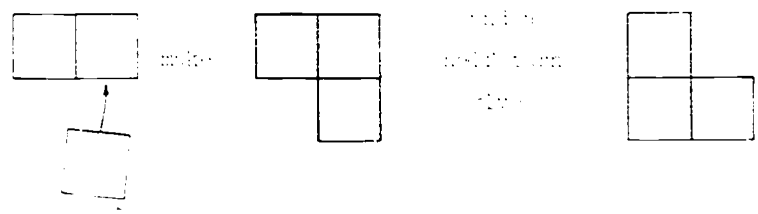


There is still another possibility:



[illegible] the first tromino and move it across the plane in such a way that it matches the second tromino. Thus, unlike dominoes, it is not true that every pair of trominoes can be matched by suitably turning and moving one of them. [illegible] this suggests the question: Can every other tromino [illegible] so that it matches one of the two trominoes shown above?

An answer [illegible] all of the possible three-square [illegible]. First, [illegible] tromino can be constructed by [illegible] of the square matches an edge of a square in the domino. If the matching edge lies at the end of the domino we get a tromino which matches the first tromino shown on previous page. If the matching edge lies along the side of the domino, then by turning and moving it across the plane it can be matched with the second tromino.

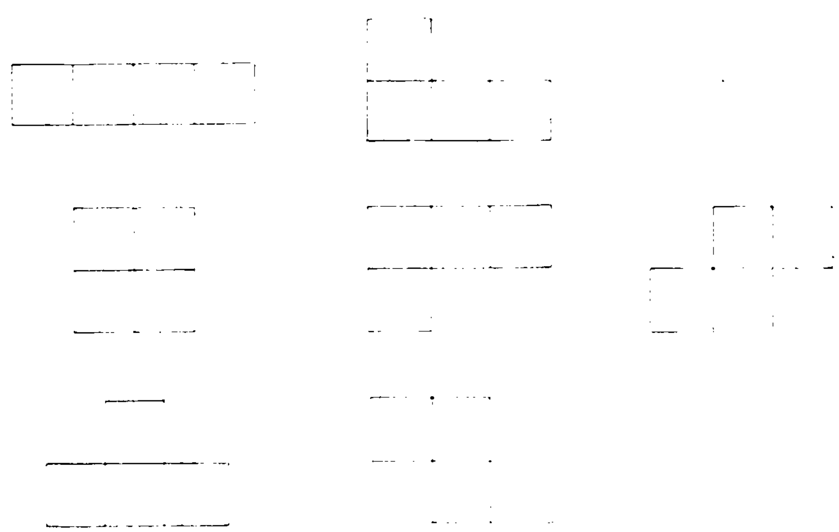


Thus there are exactly two distinct types of trominoes under motion in the plane, the *straight* tromino and the *right* tromino.

Like the domino, the straight tromino matches itself if it is given a half turn or full turn about its center. This is not true of the right tromino. It only matches itself after a full turn.

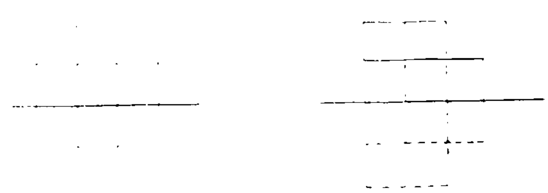
The next step in the study is, clearly, the investigation of four-square configurations. They are called *tetrominoes* or *four-squares*. As in the previous case, our aim will be to find configurations which are distinct under the motions in the plane and to describe the motions which will take a given configuration into itself.

As before, all tetrominoes can be constructed by adjoining a square to a tromino. When all of the possibilities are investigated it turns out that there are seven distinct types.

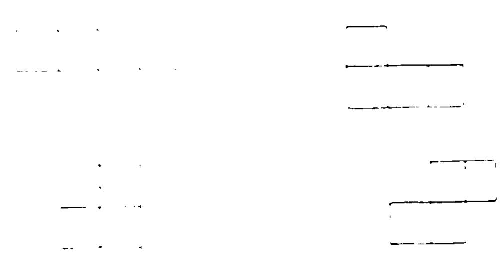


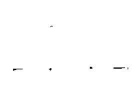
[illegible]

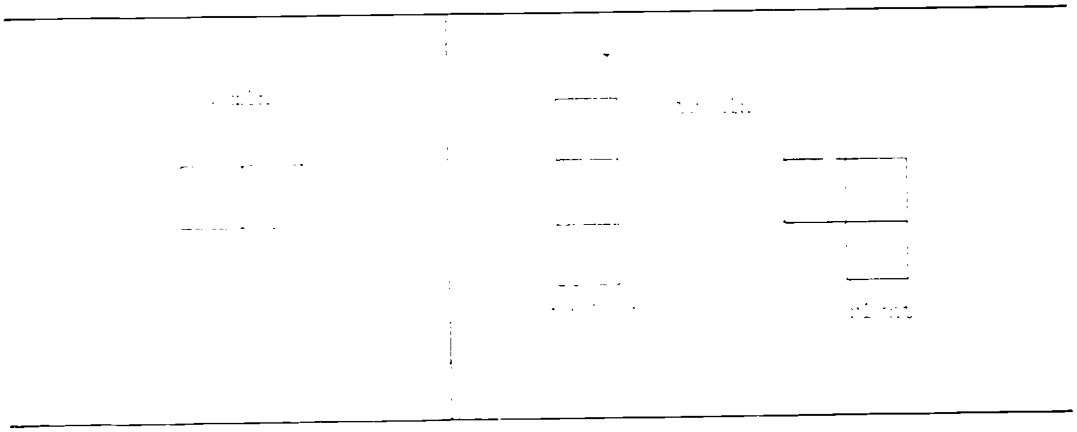
[illegible] would match the [illegible] flips it over, [illegible]

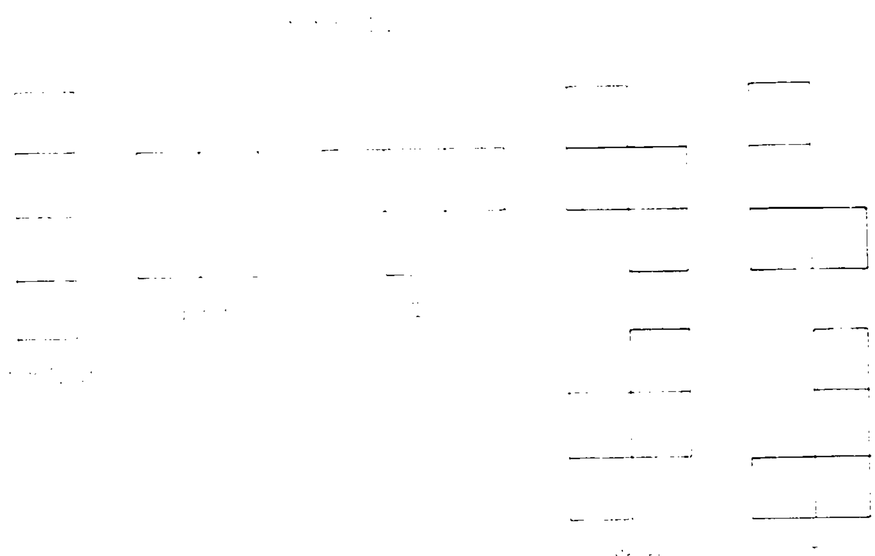


[illegible]

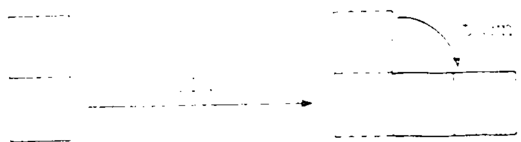








[illegible] demon-
[illegible]) and [illegible] that they
[illegible]



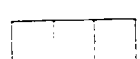
Problem

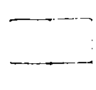
Problem: Three-square [illegible] by joining the [illegible] so that the edges match [illegible] to fit one another by [illegible]. How many different [illegible]?

[illegible] different combinations either individually [illegible] demonstrating solutions at the board or with [illegible]. They [illegible] that they must begin by forming a two-square [illegible] places to add the third square. ([illegible] ex.) Encourage them to try various possibilities and [illegible] eliminate those which can be made to fit another.

[illegible] match

[illegible] etc.



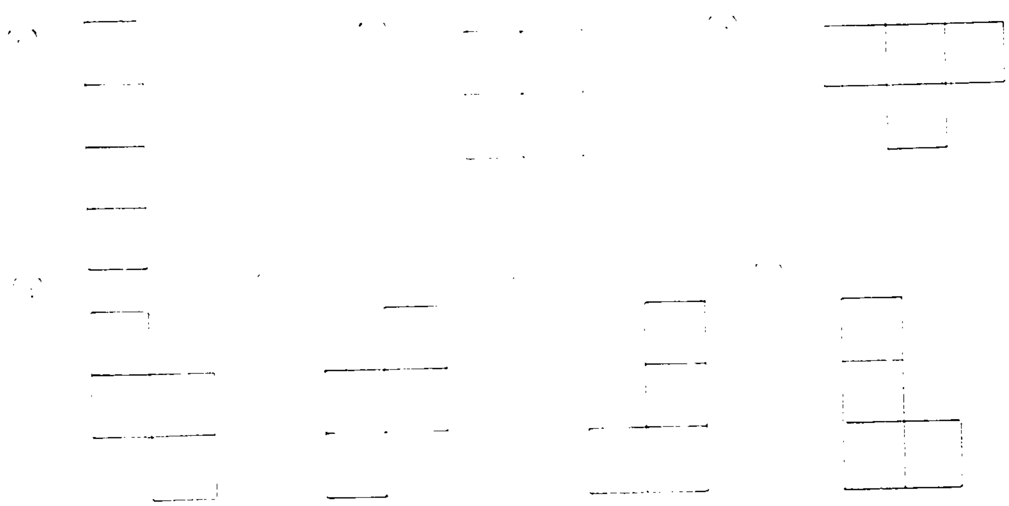


The children continue experimenting [illegible] until [illegible] three-squares.

[illegible]

[illegible]

The children [illegible] distinct shapes.



As with the three-squares, [illegible] the children to continue trying possibilities until they [illegible] all the distinct four-squares have been found. [illegible], they need to be convinced [illegible], not [illegible] that [illegible] three-squares [illegible]. Now we know that there are [illegible] distinct three-squares [illegible] consider all the places [illegible] can be joined [illegible]. The addition of a [illegible]

SPECIAL PUZZLES AND GAMES USING [illegible]

Show the children [illegible]

1. Can you form [illegible]

This puzzle will require [illegible] with the [illegible] by [illegible] [illegible] [illegible] three- square [illegible] and three right three-square [illegible].

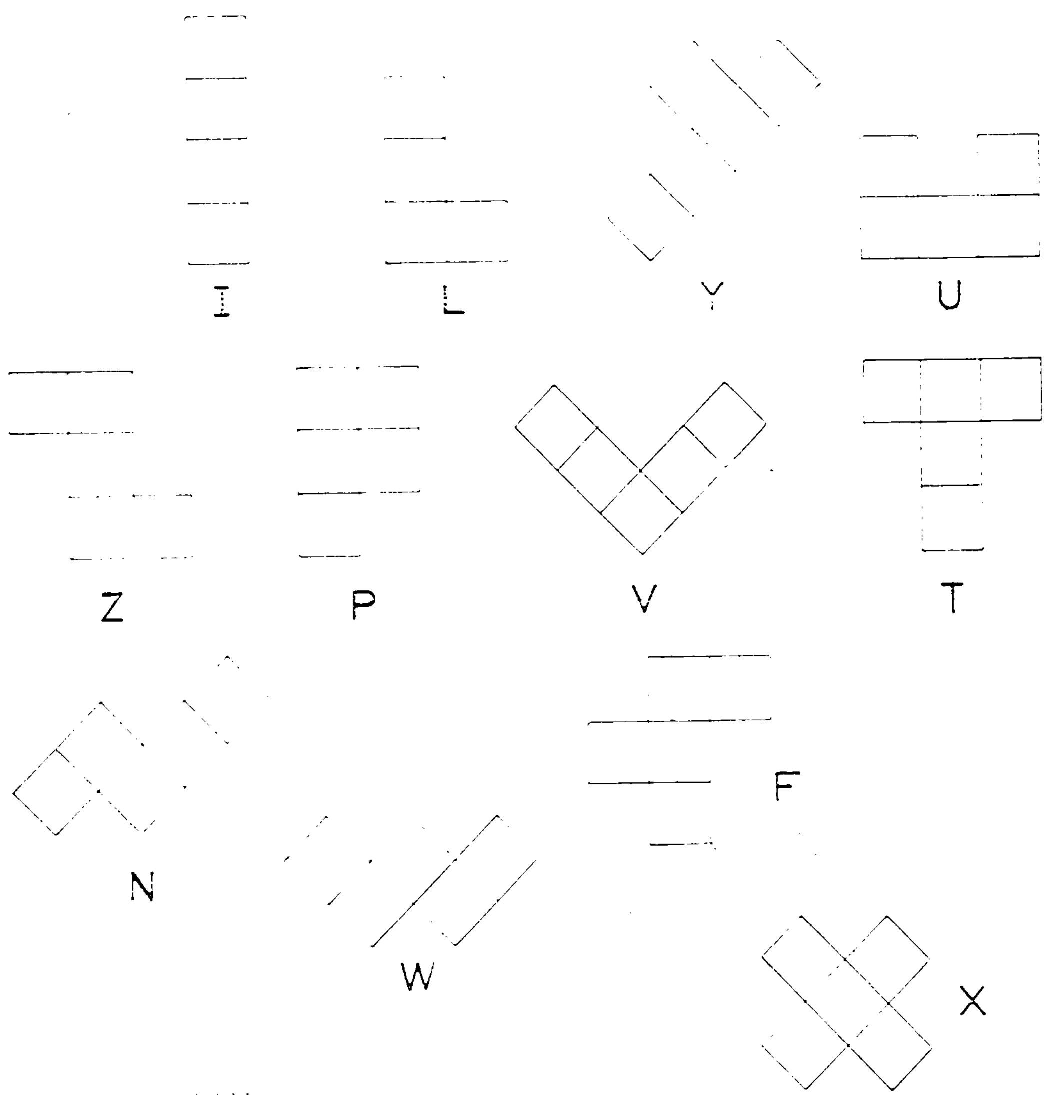
[illegible] right three-square?

[illegible] three-squares and [illegible] with two straight three- [illegible] three-square? With two right three- squares [illegible] three-square?

[illegible] Game: [illegible] two-squares. Two players take turns [illegible] on the board. The player who [illegible] wins. (The two-square must be [illegible] into two distinct [illegible]).

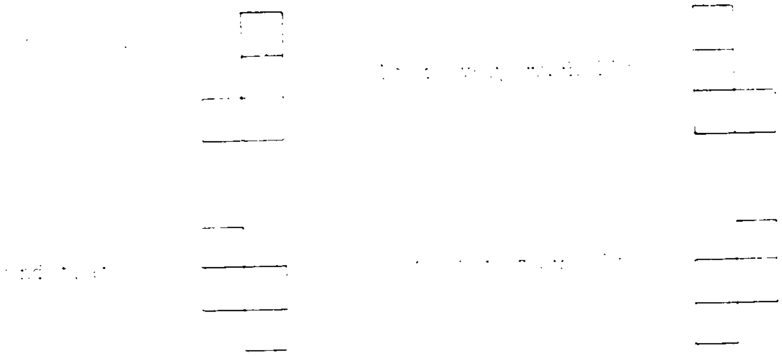
ACTIVITY II: [illegible] Slides, Turns and Flips

[illegible]



Pentominoes and Flips

[illegible] slides and turns. Ask if any pair looks similar. [illegible] the children will notice that



[illegible] to confirm [illegible] match [illegible] them. [illegible] the impossibility of fitting [illegible].

Puzzle: [illegible] move a four-square [illegible] so that it fits another?

The children experiment until [illegible] that if they flip the figures over, the first fits the second and the third fits the fourth. We now have three different motions [illegible] in matching figures, (1) slides, (2) turns and (3) flips.

Puzzle: How many different [illegible] four-squares are there if we use slides, turns and flips?

[illegible] two distinct four-squares by flips, [illegible] the number of distinct four-squares from seven to five. The children experiment with the [illegible] of four-squares by sliding, turning and flipping until they [illegible] that there are exactly five distinct shapes.

[illegible] that the number of different shapes depends on the motions [illegible]. If the rule says we can use slides and turns only, then seven different four-squares are formed; if the rule says we can use slides, turns and flips, then five different four-squares are formed.

[illegible] slides and turns, there are exactly two different three-squares (the [illegible] three-square, and the right three-square). How many different three-squares are there using slides, turns and flips?

[illegible] the eight and [illegible] five-squares cannot be made [illegible] by [illegible] However, [illegible]

[illegible]

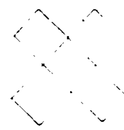
Problem: [illegible] five- [illegible] squares. Two five- [illegible]

In [illegible] children, [illegible] experiment and verify. They [illegible] then compare their results [illegible]

For example, [illegible] a configuration by [illegible] Acceptable configurations [illegible] displaying the appropriate [illegible] and continually visible to all. [illegible] it may be challenged by another [illegible] demonstrate by a combination of [illegible] already in the [illegible] five-squares are produced by the children [illegible] other possibilities. [illegible] they have not found all twelve, [illegible] think of others to try and [illegible]

1. [illegible]
squares?

[illegible]

[illegible]  [illegible]

[illegible]

[illegible] It is possible to [illegible] of shading [illegible] that [illegible] three-square [illegible] which [illegible] not adjacent [illegible]

ACTIVITY III: [illegible]

[illegible] five-square [illegible] paper with [illegible] tracing paper. [illegible] five-square [illegible]

[illegible]

Puzzle: [illegible] five-square [illegible] that it matches itself? ([illegible])

[illegible] five-square completely around in a full turn [illegible] it matches [illegible]. This can be done by the child in several ways. [illegible] piece of paper, it may [illegible] the latter turned until the traced figure [illegible]. Alternatively, the configuration [illegible]. Leaving the paper on the table, [illegible] into its original [illegible].

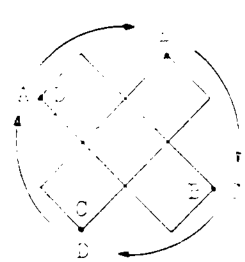
[illegible] five-square, one can trace around the [illegible] until it matches the original [illegible].

Puzzle: [illegible] five-square so that it matches itself? ([illegible])

In this [illegible] turn the figure half-way around, [illegible] original slot. [illegible] original figure. The children should discover that [illegible] to match itself under a full turn. [illegible] five-square into a matching [illegible]

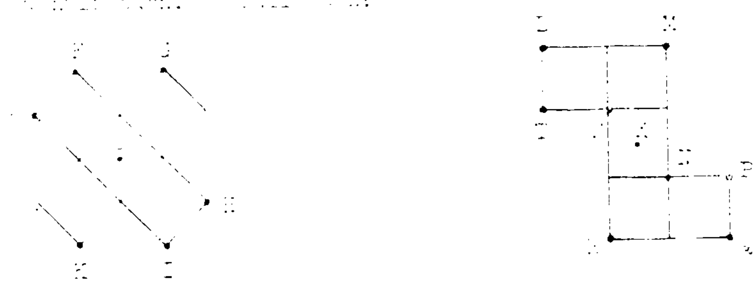
[illegible] half turn followed by another half turn [illegible].

Puzzle: [illegible] five-squares into matching figures? ([illegible])


A
B
C
D

[illegible] quarter turn [illegible], A [illegible] with F, [illegible] is matched with [illegible] with [illegible], and [illegible] matched with A.

[illegible] What happens [illegible] if you turn the figure [illegible] full turn?



[illegible] half turn, [illegible] They can record their [illegible]

[illegible]

[illegible]

[illegible] $E \to F$, $L \to L$,

[illegible]

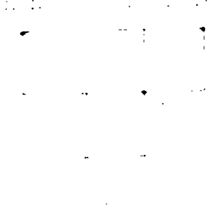
[illegible] by turning the figures. [illegible] with a quarter [illegible] (You may wish to begin [illegible])

[illegible]

[illegible] with themselves if they [illegible]

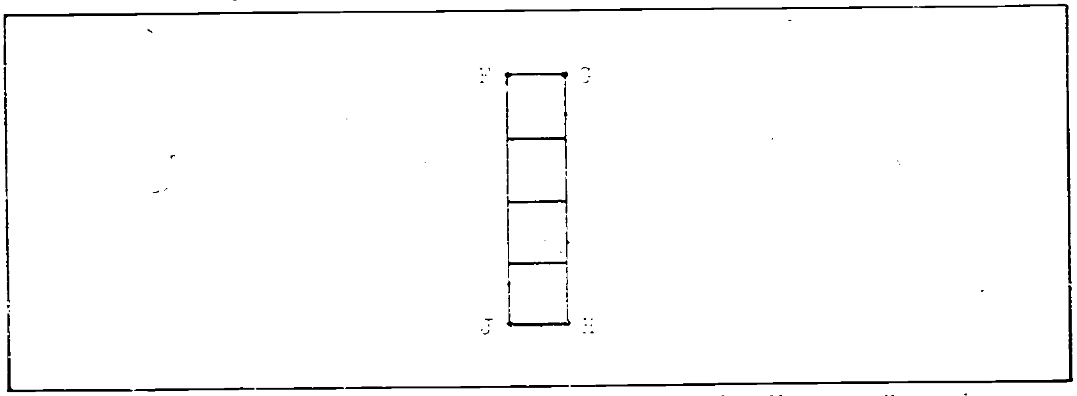
[illegible] children [illegible] flips to dis- [illegible] the original [illegible]

[illegible] flipped [illegible]



[illegible]

Puzzle: Which points match when the I five-square is flipped?



First guess the answer, then verify by flipping the figure. Here the children discover that there are two ways to flip the figure and the matching of points depends upon which way it is flipped. For example, if it is flipped about a vertical line down the center,

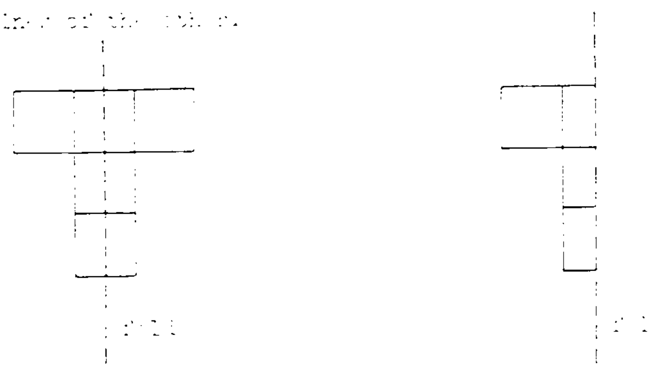
G → F, F → G, J → H, H → J.

But, if it is flipped about a horizontal line across the center,

F → J, J → F, G → H, H → G.

Puzzle: Which of the five-squares can be folded so that the lines on one side fit exactly the lines on the other side?

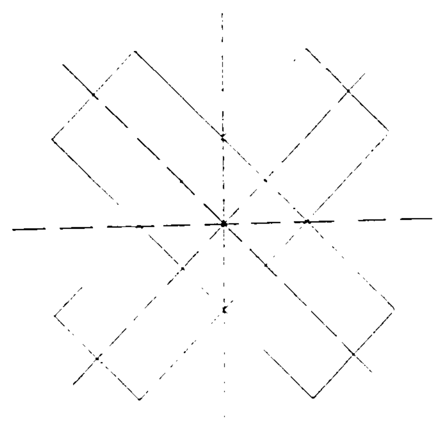
Probably figures drawn on tracing paper provide the best method of experimentation here. Demonstrate with the T five-square drawn on thin paper. If you fold down the center, the children see that the lines of one-half fit exactly the lines of the other.



Experimentation shows that the T, the I and the X can be folded so that one-half fits the other, but [illegible]. If the children compare the five-squares that could be flipped to match, they discover that those which have a folding line are exactly those which could be flipped to match. Thus a figure has a line of symmetry if and only if it can be flipped and moved into a matching configuration. The children can also discover that the I five-square has two folding lines.

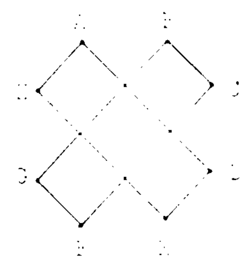
Example: How many folding lines has the [illegible] five-square?

Experimentation with folding shows that there are four lines of symmetry as shown below.



SPECIAL PUZZLER: [illegible] OF PENTOMINOES

1. Name all points [illegible] matched with [illegible] by flips.



Since there are four lines of symmetry, each flip (or fold) matches A with a different point.

A → B, or A → [illegible], or A → H, or A → [illegible]

2. Tell where [illegible] in the [illegible] are matched by a turn [illegible]:

A → ? [illegible] → ? [illegible] → ? H → ? A → A?

3. If B is matched with [illegible] (B → [illegible]), what point is matched with C?

([illegible] → A by [illegible] turn)

4. If [illegible] (H → B), what point is matched with [illegible]?

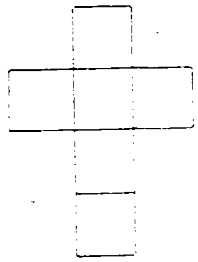
([illegible] → [illegible] by [illegible] flip)

5. However, [illegible] matched with A, [illegible]?

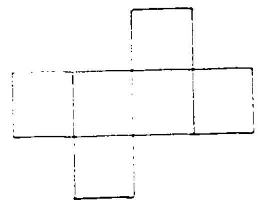
([illegible])
([illegible])

ACTIVITY IV: Some Hexominoes

Puzzle: Six-squares are made by joining six of the squares. Two examples are:



and



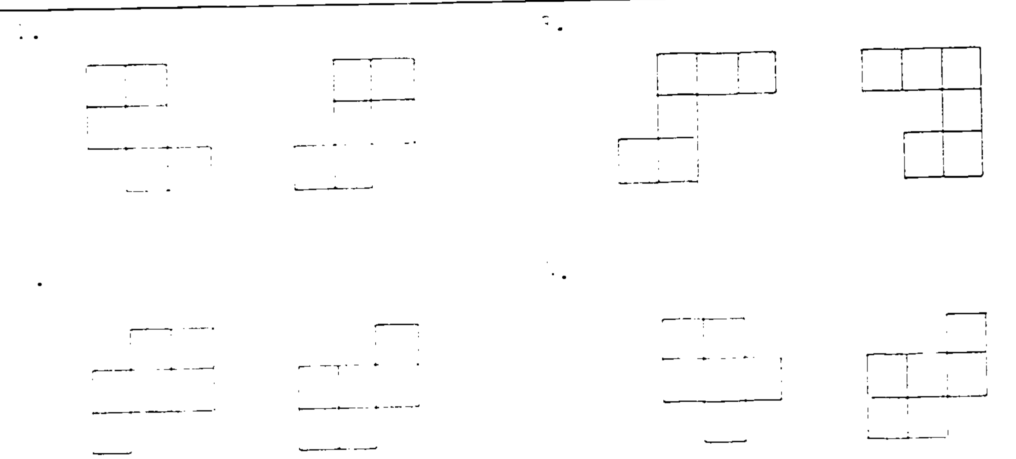
Which can be made from a four-square and a two-square?

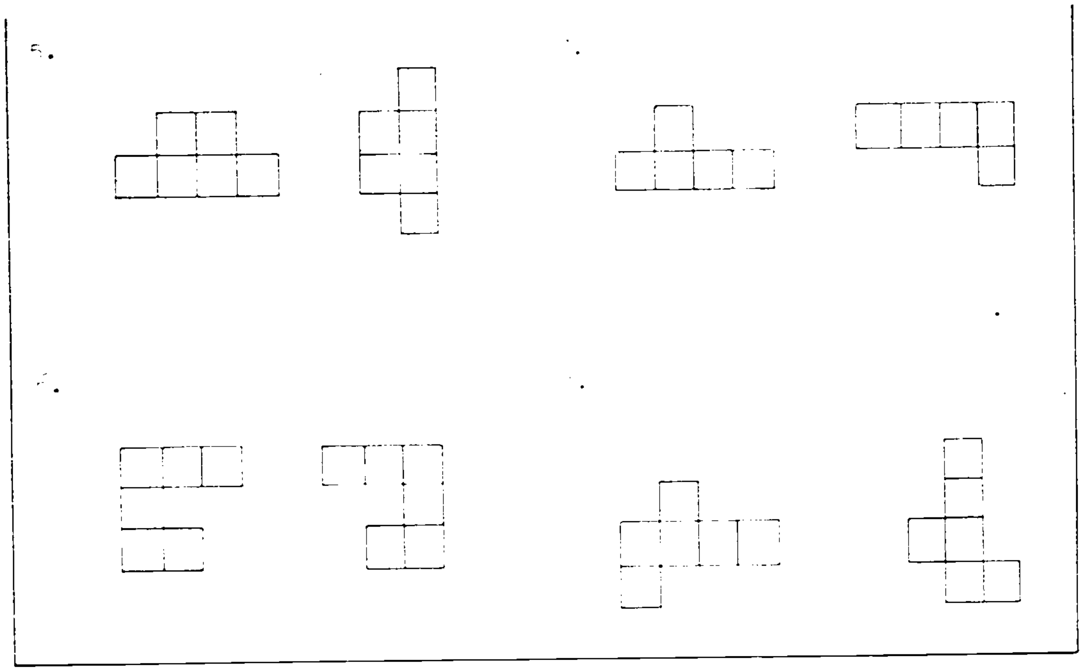
The first one, not the second one.

Puzzle: Which of the above six-squares can be made from two three-squares?

The second, not the first.

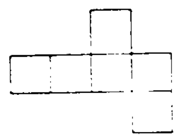
Puzzle: Which of these pairs of six-squares are different shapes and which are the same shape? If they are the same, how can you move one to fit the other?



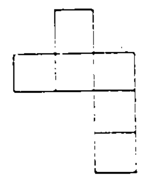


1. Which of these six-squares can be made from a four-square and a two-square but not from two three-squares?

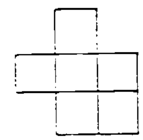
(a)



(b)



(c)



2. [illegible] 4 × 3 board. Can you completely cover the board using

(a) a five-square, a six-square and a one-square?

(b) using a three-square, a five-square and a four-square?

(c) using a six-square, a four-square, and a two-square?

(d) using a six-square, a five-square, and a two-square?

(e) using a five-square, a four-square, and a one-square?

The children may select any of the configurations listed; for example, to answer the first question they try any of the five-squares with any six-square and a one-square. They try fitting them together in the manner of a jig-saw puzzle until they find a combination that exactly fits the 4 × 3 rectangle.

This may be played as a game by two children who take turns selecting. The first child to select the three required polyominoes that will fit the 4 × 3 rectangle wins.

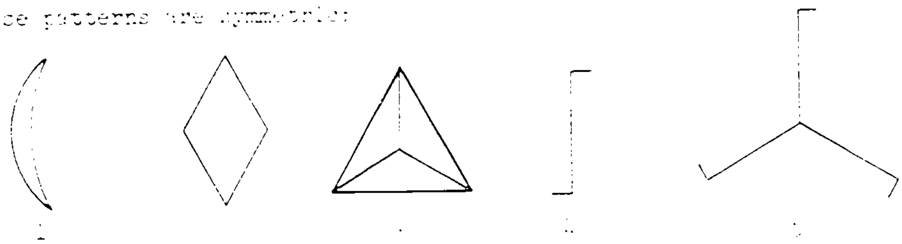
In experimenting with combinations, the child may become aware that there are 12 squares to cover. Then it will be obvious that only a combination with a sum of 12 squares could possibly fit. If he notices this, he may see immediately that the answer to (d) is "no" because the total number of squares is 13. Otherwise he may try a few combinations and finally conclude that nothing fits because there are too many squares. In either case, he may be able to apply these conclusions to question (e) and see that the answer must again be "no" because a five-square, a four-square and a one-square give a total of only 10 squares.

SYMMETRY

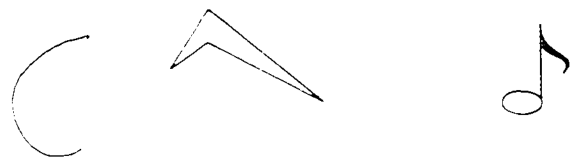
(Prepared by F. Linden)

BACKGROUND ON SYMMETRY

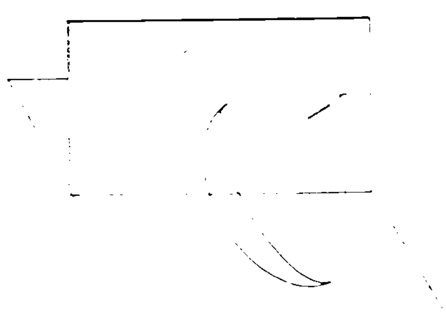
These patterns are symmetric:



These patterns are not symmetric:

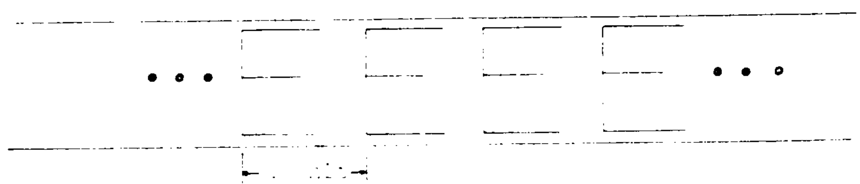


Symmetric patterns are those which look the same after being rotated, reflected or translated in some way. Patterns 2 and 4, for instance, look the same when rotated one-half turn. (To see this turn the page upside down.) Patterns 3 and 5 look the same when rotated one-third or two-thirds of a turn. Thus patterns 2, 3, 4, and 5 have rotational symmetry. Pattern 1, the lune, does not have rotational symmetry. (When rotated one-half turn, for instance, the lune looks different: the points are to the left instead of to the right.) The lune has reflective symmetry. If you place a mirror perpendicular to the paper with one edge along a horizontal line through the center of the lune as shown below:



then the visible half of the line together with its reflection looks identical to the complete line. Thus reflection is a reasonable word to describe the symmetry of the line. The horizontal line is the line's line of symmetry. The two halves of any figure folded along a line of symmetry match exactly. Pattern [illegible] has two lines of symmetry (vertical and horizontal lines through the center) and pattern [illegible] has three lines of symmetry. A square has four lines of symmetry. Patterns [illegible] and [illegible] have no lines of symmetry (though, of course, they do have centers of rotational symmetry).

The definition of symmetry given above contains three key words: rotated, reflected and translated. We have discussed the first two. A figure is said to be translated if it is shifted in some direction without changing its orientation. Only infinite patterns can look the same after a translation. Imagine, for example, that this pattern is extended infinitely in both directions:



Since this pattern looks the same if it is translated any whole number of units to the right or left, it has translational symmetry. (It also has reflection symmetry with respect to the horizontal center line.)

A second example [illegible]:

In addition to its translational symmetry, this pattern also has rotational symmetry [illegible]. (To see this, invert the page.)

[illegible] symmetric patterns.

[illegible] beyond the study of symmetry.

With [illegible] patterns, [illegible] interrelationships among symmetry [illegible] patterns [illegible] of this [illegible] patterns with [illegible] symmetry (Pattern [illegible]) [illegible] rotational symmetry. This [illegible] *the [illegible] of the lines of symmetry is [illegible] of rotational symmetry.* [illegible] to try to [illegible] lines of symmetry [illegible] symmetry.

[illegible] principle: [illegible] pattern [illegible] symmetry and no more [illegible] this is to [illegible] (Work sheet [illegible] in the [illegible] symmetry unit).) [illegible] lines must [illegible]. A pattern which [illegible] infinite [illegible] translational symmetry [illegible] translational symmetry in the [illegible] symmetry [illegible].)

[illegible] infinite patterns (wall-paper). [illegible] of Group Theory [illegible] symmetry in [illegible] infinite patterns. [illegible] experimentation. [illegible] in the [illegible].

[illegible] symmetry is important [illegible] for introducing the [illegible] that symmetry and abstract [illegible] mathematics. In [illegible], symmetry [illegible] the [illegible] anything [illegible] high school geometry [illegible] construction [illegible] way this [illegible].

[illegible] symmetry [illegible].

References

1. For ingenious art based on symmetry see: M. C. Escher, "The Graphic Work of M. C. Escher," Meredith Press, New York, 1967.

. For mathematical theory with many pictures see: L. Fejes Tóth, "Regular Figures," Pergamon Press, New York, 1964.

SYMMETRY UNIT

(1) Definition by Example

Display a collection of symmetrical shapes (labeled "symmetrical") and a collection of unsymmetrical shapes (labeled "unsymmetrical"). (Suggested shapes are attached.) After the students have studied the two collections for a few minutes produce new shapes [illegible] to be identified as symmetrical or unsymmetrical. Do not try to define symmetry in words. Include many kinds of symmetry (stars, [illegible], [illegible], etc.). [illegible]

(2) Meaning of Symmetry (transformations)

Display a symmetrical shape. Stand in front of it and pretend to manipulate it. [illegible] "[illegible]." Since the shape is symmetrical, [illegible]. This serves as a definition: If a shape can be moved in some way without changing its appearance, it is symmetrical.

[illegible] including [illegible] rotational symmetry. A symmetrical [illegible] than one way, [illegible].

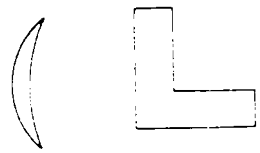
(3) Lines of Symmetry

[illegible] line so that [illegible] of the [illegible]. [illegible] line of symmetry (fold line) [illegible] (normal letters, swastika).

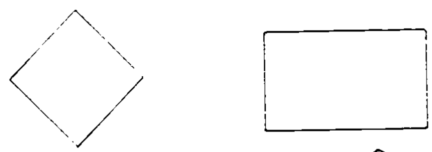
F SUGGESTED SHAP

No symmetry:

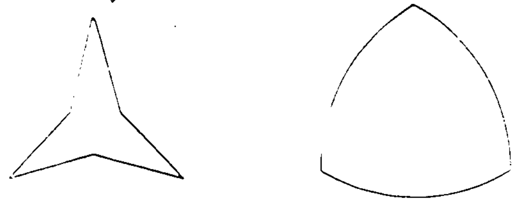
One line of symmetry:



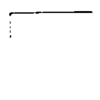
Two lines of symmetry:



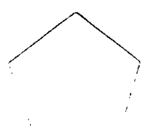
Three lines of symmetry:



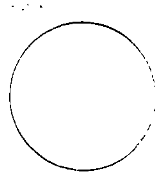
Four lines of symmetry:



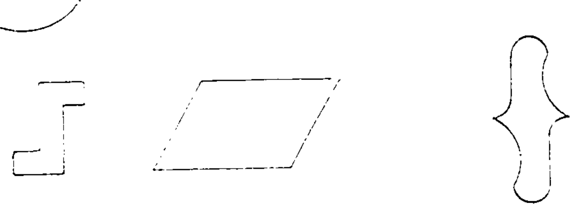
Five lines of symmetry:



∞ lines of symmetry:



-Fold rotation:



4-Fold rotation:

(Mirror mounted on block.) Observe that if a mirror is placed along a line of symmetry, the visible half of the figure together with its reflection looks the same as the figure itself.

Observe that shapes with a line of symmetry fit into their outlines when flipped over. (Shapes with rotational symmetry fit into their outlines when turned like a pinwheel.) You have now tested for lines of symmetry three ways: (1) by folding, (2) by reflecting in a mirror and (3) by flipping. Establish in discussion that the three tests are equivalent. A good way to do this is to challenge the pupils to find a shape that passes one test (e.g., folding) but fails another (e.g., reflecting). Of course, this is impossible. Do some shapes have more than one line of symmetry?

(4) Work Sheet 1 (attached)

Pass out work sheet 1 for pupils to do in class. Go around class with mirror checking worksheet 1. Demonstrate errors by placing mirror along erroneous lines of symmetry.

(5) Homework

Find (capital, block) letters of the alphabet with,

(a) no symmetry.

(b) one line of symmetry.

(c) two lines of symmetry.

(d) rotational symmetry but no line of symmetry.

(6) Game

Equipment: A [illegible] grid of squares divided by a heavy line into two [illegible] halves, [illegible] cardboard squares each of such size as to cover exactly four squares of the grid.

Rules: Players take turns placing squares on grid. Each square must cover exactly four grid squares. Once a square is put down it [illegible].

No square may [illegible] the heavy center line, and no square may overlap another square which has been put down before, and no square may extend off the grid.

A player wins when the opposing player cannot find a place to put his square.

[illegible] the hidden figure.

[illegible] symmetry in the [illegible] symmetry [illegible].

[illegible] of figures [illegible] on the side [illegible] more, [illegible].

[illegible] in [illegible] mathematical ideas.

[illegible]

[illegible]

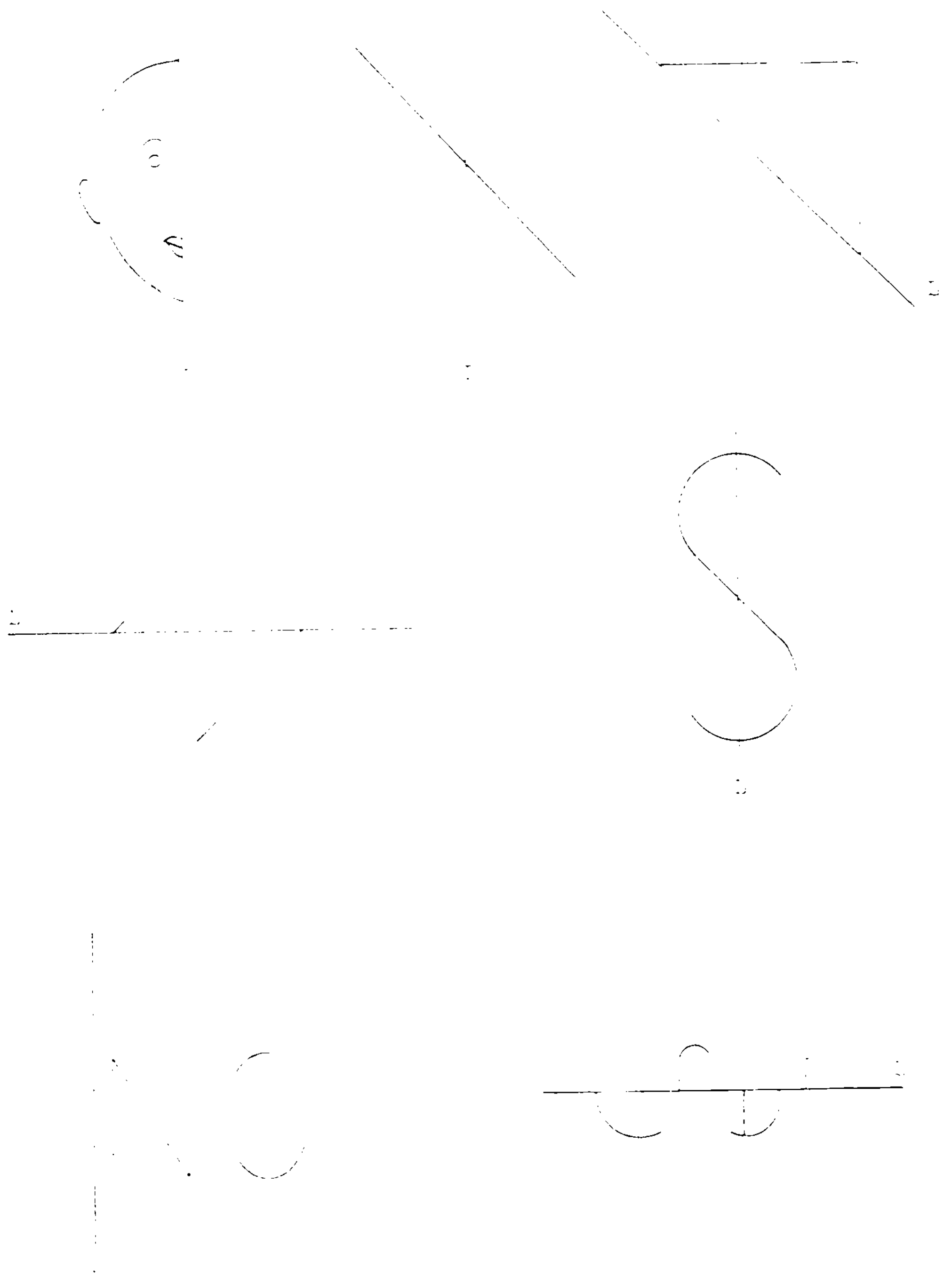
[illegible] but no line of symmetry. [illegible] like a pinwheel. Dis- [illegible] ... [illegible] symmetry. Now find rotation- [illegible] of symmetry. Count the lines of [illegible] more than one line of [illegible] and only one line of [illegible] with -fold rotational [illegible]. When they are convinced it is [illegible].

[illegible]

[illegible]

WORK SHEET [illegible]

Draw all lines of symmetry.



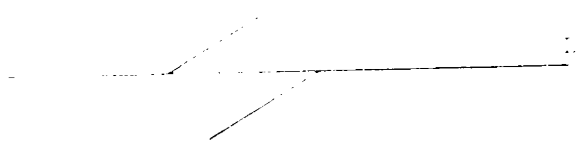
[illegible]

[illegible] . [illegible] pr[illegible] . [illegible]

[illegible] . Most did not [illegible] symmetric [illegible] look [illegible] mirror which [illegible] surprised [illegible] to preserve [illegible] in item 3. [illegible] the figure flat and shows [illegible] standard terms, [illegible]."

[illegible]: [illegible] of symmetry for the heart [illegible]. [illegible] line on the [illegible] mirror transform the [illegible] vertical and horizontal [illegible] but failed to find the [illegible]. Similarly, most found only one of the [illegible] triangle. A [illegible] work out right without [illegible] but it should be [illegible].

[illegible]: [illegible]. Most were puzzled by the [illegible] horizontal line, some even [illegible]. [illegible] been replaced by:



[illegible] very exactly. I'm [illegible] around. About half got [illegible]. [illegible] one was harder [illegible].

Game: The game was played on a Vu-graph, but a flannelboard would be [illegible]. It generated a lot of excitement and noise. I started out playing the [illegible] strategy (rotation instead of reflection) and switched strategies [illegible] time. This precaution was unnecessary. Although the students quickly [illegible] it, [illegible] the game [illegible]. It would be better [illegible] (similar) [illegible] reflection symmetry [illegible] until they catch on. Then play other strategies (i.e., other transformations). They found out soon [illegible] that it is [illegible] to [illegible] second. It [illegible] winning [illegible] if you go first and your opponent fails to play symmetry strategy, but it can usually be done. Your winning demonstrates [illegible] to the student that he has not yet found [illegible].

Even after the students knew how to win, it took considerable [illegible] understand clearly why the symmetry strategy works. Probably less than half understood it at the end. The reasoning [illegible] follows: If the pattern of squares is symmetric then [illegible]. Therefore, if there [illegible], there must also be a space (on the other side) [illegible].

[illegible]

[illegible] that this unit puts unusual demands on the [illegible] unfamiliar. The material will be successful only if the teacher [illegible] himself conscientiously. In particular, he should work through all the questions and problems in detail and study the [illegible].

[illegible] a few specific observations.

The mirrors [illegible], especially in showing up errors on the worksheets. [illegible] the effect is dramatic and convincing. [illegible] down on such occasions.

[illegible] reflection symmetry. Many children and some teachers seem to do quite a bit of folding and [illegible] before they are convinced that it has no line of symmetry.

[illegible] to fit the shape [illegible] should be drawn beforehand. It is too [illegible]

[illegible] end view. [illegible]

[illegible] ACTIVITIES BASED ON COMPUTER [illegible]

A [illegible] MACHINE

(Prepared by W. [illegible] and [illegible])

INTRODUCTION

The [illegible] of [illegible] activity [illegible] students [illegible] certain aspects of number representation [illegible] -like classroom [illegible] major phases to [illegible]:

1. The [illegible] parts of [illegible] counting machine [illegible]. In this phase (which should take 5-10 minutes), the emphasis [illegible] getting the "machine" to perform correctly [illegible] the class to identify the relation between the number [illegible] and the [illegible] of the "machine parts."

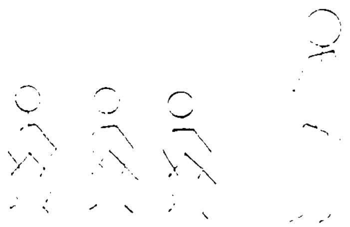
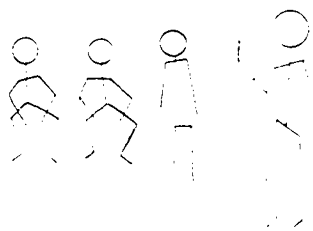
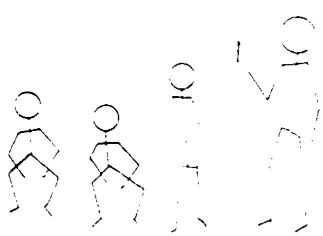
2. When [illegible] fairly [illegible], the teacher asks [illegible] identify [illegible] number for any machine [illegible] representation for any number (if it [illegible]. This [illegible] about 5-10 minutes.

3. Finally, [illegible] the students can answer by [illegible] machine [illegible]. However, the questions can also be [illegible] relationship between the representations [illegible] representation of numbers. [illegible] for developing the kind of [illegible].

[illegible]

[illegible], [illegible] numbers. When we write [illegible] of hundreds, [illegible] to represent [illegible] four [illegible]. We can emphasize this by writing [illegible] $100) + (5 \times 10) + (4 \times 1)$ or [illegible]. [illegible], this representation of numbers [illegible] represent a number such as one thousand, one hundred, [illegible] another 1 and write 1,154. The 1 [illegible] of thousands or 1×1000 or [illegible].

ACTIVITY: In 1

Teacher says:	Classroom arrangement:	Comments:
"Has anyone ever seen a counting machine?"		Elicit students' experiences pointing out mechanical nature and forms of suggested counting machines.
"Today we are going to make our own counting machine. In fact, the moving parts of our counting machine will be some of you."		Select three students and arrange in three chairs at front of room.
"Now let's see how the parts of our machine will move when it is counting." "Now the machine will count to one."		Write a tally mark on the board. Pass the chalk to the student seated in the first chair. Instruct the student to stand and pass the chalk to the student in the second chair. Instruct each student to pass the chalk until you get it from the student in the third chair.
"This is the way the machine shows the number one."		It is important to establish the relationship between each number counted as recorded in the tally and the machine representation of each number.
"Now the machine will count to two."		Before counting to two, be sure that no one has changed positions since you counted to one. Write a tally mark on the board. Pass the chalk to the student standing at the first chair. Instruct him to change positions and pass the chalk to the second student.

Teacher Says: | [illegible] | Comments:

He instructs the second student to change position and stand. The third student once again [illegible] the chalk, remains seated, and passes the chalk to the teacher.

"This is the way the machine counts the number two."

"Now the machine will count to three."

Repeat the same procedure for one more tally.

"This is the way the machine counts the number three."

"This is the way the machine operates. I [illegible] the board with a piece of chalk. This is the number which is to be counted. I pass the chalk to the student in the first chair. He always changes his position and passes the chalk to the student in the next chair. [illegible] and all other machine parts simply watch the [illegible] passed [illegible]. If the student [illegible] the chalk [illegible], the next student changes his position and passes the chalk."

After instructing the machine for several tallies, then the rules for the motion of each part should be given.

As you can see the sitting down of a student is simply a signal for the next student to change position. This sitting down is like the carrying which takes place in adding.

"If the student [illegible] the chalk [illegible] [illegible]"

It acts in the same way as the mileage indicator on a car; when the right hand dial moves from nine to zero, the next dial is advanced one digit.

Teacher says:

"Let's use the [illegible] for each machine part and count to four."

"This is the way a machine [illegible] number four."

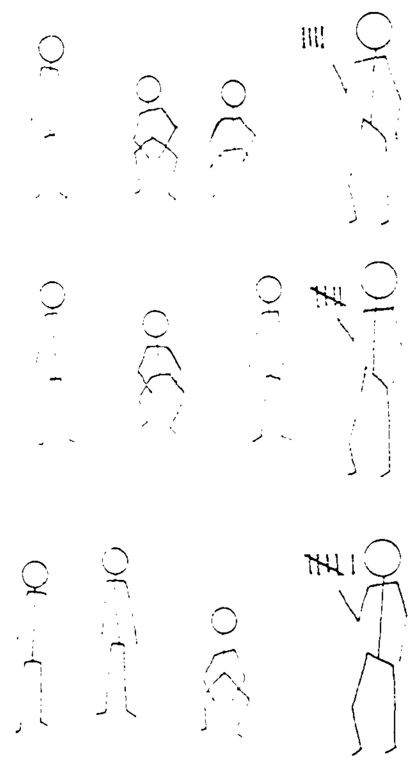
"Now let's count to five."

"This is the way the machine [illegible] number five."

"And now six."

"This is the way the machine [illegible] the number six."

[illegible]:



Comments:

Follow the [illegible] in [illegible] tallies. [illegible] machine [illegible] moving [illegible] for the [illegible] machine's [illegible]. This is also an excellent way to [illegible] in the process. You may want to [illegible] the machine parts after you have succeeded in getting one machine to work properly. A second machine will ordinarily work properly much more quickly. In selecting the machine parts, it is necessary that the first student be able to keep the operation moving. Each machine part is important, but smooth counting machine operation is lost when the first machine part does not respond to the tallies.

ACTIVITY: Phase I

Teacher Talk:

"Now let's see if we can tell what number the machine is naming. What number is the machine naming when it looks like this?"

"What about this?"

"What number is the machine naming when both these machine parts are standing?"

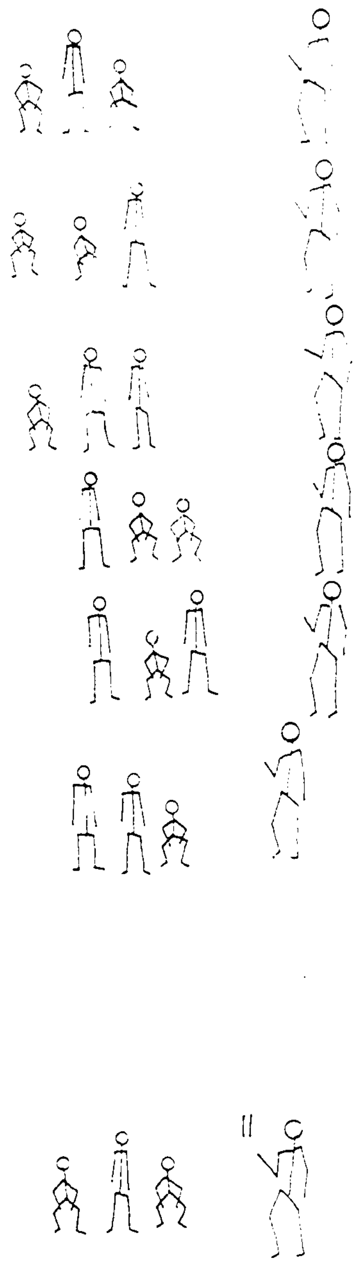
"What number is this?"

"And now this?"

"And this?"

"Now let's see how the machine will name a number we pick. Let's try two. Who would be standing?"

Classroom Apparatus:



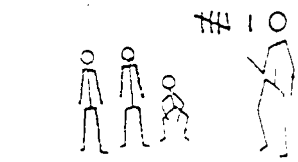
Comments:

You now control the parts of the machine to make the representations you want. If the students in the class are not able to identify the representations either by memory or by using the base two, you can always begin the counting again from zero until you get to the representation you want. Do not attempt to explain the relationships. Give the students an opportunity to discover whatever relationships they can for themselves by your careful selection of examples. Note that the suggested order in which the representations are presented emphasizes that two is the base and that the representations of each machine part are added.

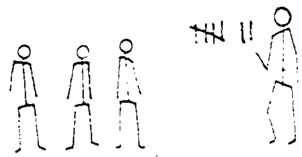
The students in the class should be able to respond to these with little difficulty.

Teacher says: | Classroom application: | Comments:

"Let's try naming six on the machine. Who should be standing?"



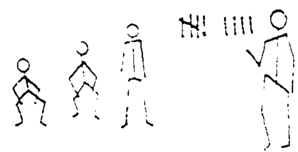
"Who should be standing if we name seven on the machine?"



You might let the class suggest numbers to represent on the machine. If you do this, and any number greater than seven is mentioned, you will be ready to move to the Phase 2 activity.

ACTIVITY: Phase 2

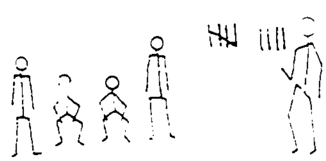
"How would the machine name the number nine?"



"This is not very satisfactory since this is also the way the machine named ten. What could we do to the machine in order to be able to name nine in a different way than ten?"

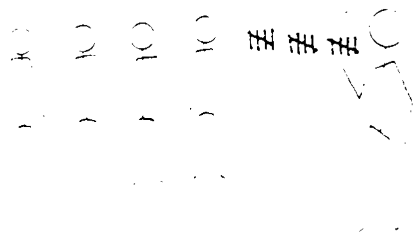
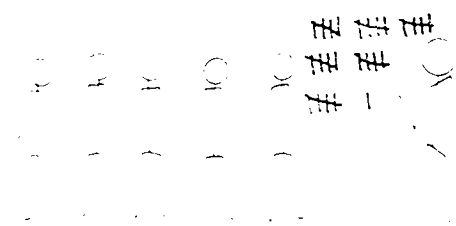
You are now ready to ask some of the questions which involve the students in extending their observations to new situations with the counting machine. Since the students are not likely to be able to answer these questions immediately, you should begin with a number representation they know -- such as seven -- and count to nine using the rules.

"Let's add another machine part."



"Now let's count from seven."

The students will ordinarily suggest that if another machine part is added to the machine the number nine can be named in a different way than ten. Add another student as a machine part and begin counting from seven again.

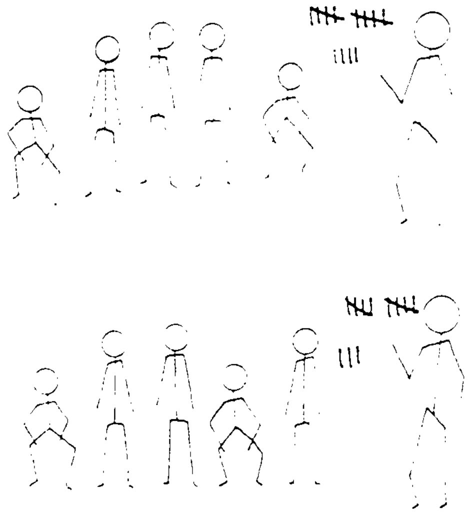
Teacher Says:	Classroom Happenings:	Comments:
[illegible]		Here the students should be given a chance to guess. The correct answer is fifteen. You can handle the responses in a variety of ways. You can count and check the responses by simply following the rules for the machine, or you can examine the representations for each machine part.
		You can now return to the Phase [illegible] type activity and ask for representations of numbers on the machine and for the numbers represented by the machine with the additional machine part.
[illegible]		Additional machine parts may be added and the many possibilities explored. It is impossible to predict all the directions which this activity will take. You should be open to the variety of responses from the students so that they will be able to freely explore the relationships involved in the binary representation of the numbers on the machine.

Teacher Talk: | Classroom Situation: | Comments:

"Now let's name the number fourteen on the machine."

"Change the rules for the machine so that each part after the first one changes position when the machine part who passes the chalk to him stands up, instead of when he sits down."

"What number is named by the machine now?"



One interesting change in the rules might be explored. If each machine part after the first one changed position only when the machine part who passes the chalk to him stands up, we have a reversal of the counting procedure. This new rule enables us to count backwards.

[illegible] end. The counting machine [illegible] -- complex [illegible] simpler ones. This is important [illegible]. In addition, in order to get the counting machine [illegible] work is involved. [illegible] structure and [illegible] work. Furthermore, [illegible] preparation for the eventual study [illegible] some high school students.

[illegible] activity. The activity [illegible] experienced in [illegible]. First [illegible].

[illegible] children [illegible] com- [illegible]:

1. [illegible]

2. [illegible]

3. [illegible]

APPRAISAL

[illegible] students about whom you are unsure as to whether they have acquired the [illegible] described in the objectives.

(Obj. 1) 1. Arrange [illegible] machine so that it represents the number four. Ask each student individually, "What number is named by the machine?"

(Obj. 2) 2. Arrange the machine so that it represents the number eleven. Ask each student individually, "What number is named by the machine?"

(Obj. 3) 3. Make available a [illegible] machine with four machine parts. Ask each student individually, "Arrange the machine so that the machine names the number [illegible]."

(Obj. 4) 4. Make available a [illegible] machine with four machine parts. Ask each student individually, "Arrange the machine so that the machine names the number [illegible]."

(Obj. 5) 5. Make available a [illegible] machine with four new and inexperienced machine parts. Write the numeral [illegible] on a sheet of paper [illegible] each student. [illegible] each student individually, [illegible] the machine [illegible] so that the machine names [illegible]."

CLASSROOM ACTIVITIES BASED ON COMPUTER CONCEPTS

FINDING THE GREATEST COMMON DIVISOR

(Prepared by W. [illegible] and J. [illegible])

After you have read the following few pages, you will be able to organize a demonstration of a computer finding the greatest common divisor of two numbers using the children in your classroom as the participants.

GREATEST COMMON DIVISOR

But before we begin, let's find the greatest common divisor of two numbers -- sometimes known as the greatest common factor of two numbers. This will not be the procedure that will be followed in the computer demonstration, but it will provide good background for us. We know that [illegible] is a factor of [illegible] in the set of whole numbers because there is a whole number [illegible] such that [illegible] [illegible] is not a factor of [illegible] because there is no whole number which when multiplied by [illegible] [illegible].

Now we can write all of the factors of [illegible]. List them.

____.

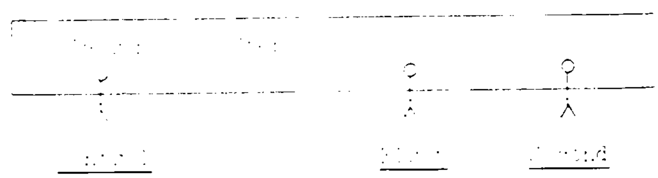
Your list should include 1, [illegible], [illegible], [illegible], [illegible], and [illegible]. These are all of the factors of [illegible]. Can you write the set of all factors of 18? __________

__________. Your answer should be {1, [illegible], [illegible], [illegible], 9, 18}. What factors of [illegible] and [illegible] are the same? __________. 1, [illegible], 3, and [illegible] are the factors which are the same for [illegible] and 18. These are called the common factors. [illegible] [illegible] of these numbers is [illegible]. [illegible] is known as the [illegible] common factor or the greatest common divisor. Let's find another greatest common divisor.

[illegible] the greatest common divisor of [illegible] and [illegible] __________. The answer [illegible] [illegible] {1, 3, 5, 15} and of [illegible] is {1, [illegible]}. [illegible] only common [illegible] of 15 and [illegible], and [illegible] greatest common divisor of 15 and [illegible].

[illegible]

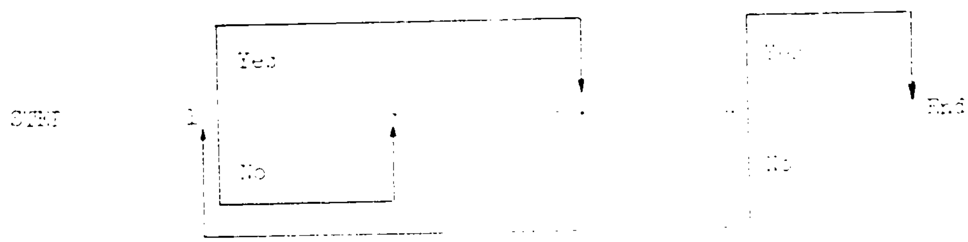
[illegible] in order to find the greatest common divisor [illegible] ways. [illegible] -- [illegible], [illegible], and Second -- in the [illegible]:



[illegible] that appear on the [illegible]. [illegible] only four [illegible]. [illegible] pair for which the [illegible] and [illegible] writes down the second of the [illegible] common divisor of 1[illegible] and 18 with [illegible]. [illegible] and Second writes "1[illegible]."

[illegible] and 15, the [illegible] following the arrows [illegible]. [illegible] the number of [illegible].

DIAGRAM FOR FINDING THE GREATEST COMMON DIVISOR

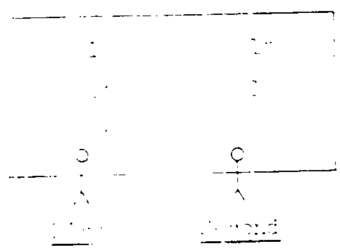


STEPS FOR FINDING THE GREATEST COMMON DIVISOR

STEP	STUDENT	ACTION
1	First	Calls "Yes" if his number is greater than or equal to Second's number, "No" if smaller.
[illegible]	First and Second	Trade numbers -- First writes down Second's number and Second writes down First's number.
[illegible]	First	Subtracts Second's number from his and writes the difference as his number.
[illegible]	First	Calls "Yes" if his number is 0. "No" if it is not 0.

Control [illegible] call out the name of the student -- First or Second -- who is involved in [illegible] in that step. [illegible] send the action for that student for the step. [illegible] follow the activity with [illegible] and 1[illegible].

Control points to step 1 in the diagram and then says, "First," since [illegible] First [illegible] active in step 1. Since First's number is smaller than [illegible], First says, "No." Control then follows the arrow [illegible] for the [illegible] in step 1 on the diagram and finds he is pointing to st[illegible] the diagram. Control says, "First and Second," since First and Second [illegible] active in step [illegible]. First [illegible] writes down [illegible], and Second writes down [illegible] in order to change the numbers [illegible]. Control then [illegible] to step 3 in the diagram and [illegible], "First." [illegible] Second's number from his and writes the [illegible]. [illegible] First and Second would look like this [illegible]:



[illegible] and says, "First." First calls, [illegible]. Control then follows the arrow [illegible] step [illegible] and says, "First." And [illegible] and [illegible] the two numbers.

[illegible] following table:

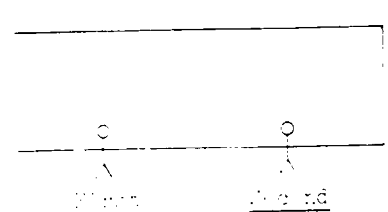
Control [illegible]:	[illegible]	Second writes [illegible]:	Step:
		[illegible]	Begin
"[illegible]"	[illegible]		1
[illegible]	[illegible]	1	[illegible]
"[illegible]"			[illegible]
"[illegible]"	[illegible]		[illegible]
[illegible]			1

Can you now begin to fill in the remaining responses in the table? What does *First* do at the point we stopped in the activity back at step 1? Does he call, "Yes" or "No"? ________. He must call, "No," since his number 6 is smaller than *Second's* number [illegible]. Try completing the rest of the steps and compare with the completed table [illegible].

Control calls:	*First* writes or calls:	*Second* writes or calls:	Step:
-	[illegible]	[illegible]	Begin
"*First*"	"No"		1
"*First* and *Second*"	[illegible]	1	[illegible]
"*First*"	[illegible]		3
"*First*"	"No"		4
"*First*"	"No"		1
"*First* and *Second*"	[illegible]	[illegible]	[illegible]
"*First*"	[illegible]		3
"*First*"	[illegible]		4
* "*First*"	[illegible]		1
"*First*"	[illegible]		3
**"*First*"	[illegible]		4
			End

The last number that *Second* has written down is the greatest common divisor. As we already know from [illegible] earlier examination, [illegible] is the greatest common divisor of [illegible] and 1[illegible].

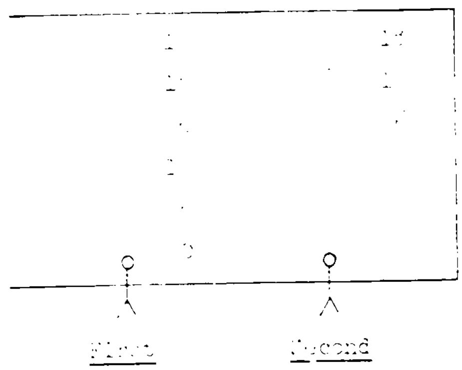
What would *First* and *Second* have written on the chalkboard in front of them in finding the greatest common divisor of [illegible] and 15? See if you can fill in the [illegible] on the chalkboard below:



*At this point in the activity *First* says, "[illegible]" since [illegible] is not smaller than [illegible]. Then the activity [illegible] following the arrow skips step [illegible].

**At this point in the activity *First* [illegible] since the difference is [illegible]. [illegible]

It should look like this:



Now try one on your own. Write out the table of responses for finding the greatest common divisor of [illegible] and [illegible].

Control calls:	First writes or calls:	Second writes or calls:	Step:
		[illegible]	Begin

[illegible] common divisor of [illegible] and [illegible]

You can now [illegible] your class [illegible] in determining the greatest common divisor. This activity [illegible] the class [illegible] of the aspects of computer [illegible]. [illegible] complex procedure is [illegible] by the repeated use of very simple [illegible]. In addition, good team work is necessary. [illegible] of simple computer [illegible].

CLASSROOM [illegible]

[illegible] introduced in the classroom. If [illegible] already [illegible] find the greatest common divisor of [illegible] numbers, [illegible] to this. If students have not [illegible], the activity [illegible] introduced [illegible] the building of a computer [illegible] find the number [illegible] simplify a fraction such as [illegible]. [illegible]. Then, the fraction can be rewritten as [illegible]. [illegible] is provided for [illegible], it [illegible] appropriate to also [illegible]: "[illegible]", "Can computers [illegible] complicated problems?"

[illegible] will [illegible] take [illegible] while the [illegible] and [illegible] the steps. [illegible].

[illegible] in advance. If [illegible].

[illegible] by the computer [illegible].

[illegible].

[illegible] and [illegible] each group [illegible] compute [illegible] the board [illegible]
[illegible] *[illegible]* [illegible] group
of subtraction.

Naturally, the choice of pairs of numbers [illegible] to find the [illegible]. Pairs [illegible] involve three subtractions. The pair [illegible], and the pair 35, [illegible] involves eleven subtractions. The pair [illegible] the greatest common divisor while the pair [illegible] greatest common divisor. Incidentally, the pair [illegible], [illegible] involves only two subtractions. [illegible] The examples you choose will depend upon what [illegible].

[illegible]

[illegible] in your classroom [illegible] common divisor of two numbers:

1. [illegible].
2. [illegible] common divisor.
3. [illegible].
4. [illegible].
5. [illegible] common divisor.
6. [illegible] on board.
7. [illegible].
8. [illegible].
9. [illegible].
10. [illegible] over [illegible].

[illegible]

[illegible]

1. [illegible] greatest [illegible]

. The student will demonstrate the operation of the computer by playing the role of Control in the computer activity.

Naturally, one of the [illegible] would be the expectation that students will play the role of Control in another computer program activity. [illegible], the student will [illegible] procedures in mathematics.

APPRAISAL

[illegible] if the students have acquired the immediate behaviors for the activity [illegible] by asking the following questions. [illegible] all students [illegible] an exercise to be written [illegible] demonstration with students.

Obj. [illegible]. "[illegible] of the computer, can you find the greatest common divisor of [illegible] and [illegible]?"

Obj. [illegible]. "[illegible] method of the computer, can you find the greatest common divisor of [illegible] and [illegible]?"

Obj. [illegible]. [illegible] Control and [illegible]. [illegible] the student to [illegible]. "[illegible] demonstrate how you would [illegible] in finding the greatest common divisor of [illegible] and [illegible]."

LINEAR FUNCTION GAMES

(Prepared by W. [illegible] and [illegible])

GENERAL DESCRIPTION OF VARIOUS LINEAR FUNCTION GAMES

A variety of classroom activities can be devised in which all pupils can participate either as simple arithmetic "machines" or as [illegible] of the operation of "assemblies" of such machines. The [illegible] of games described below use the basic arithmetic transformations like "multiply by 3" or "add 4" as simple machines.

The fundamental elements of these games are the following:

1. A pupil, the machine, selects, or is assigned by the teacher, one of an agreed upon set of rules for transforming or converting a number into another number. The rule might be, for example, "multiply by 3."

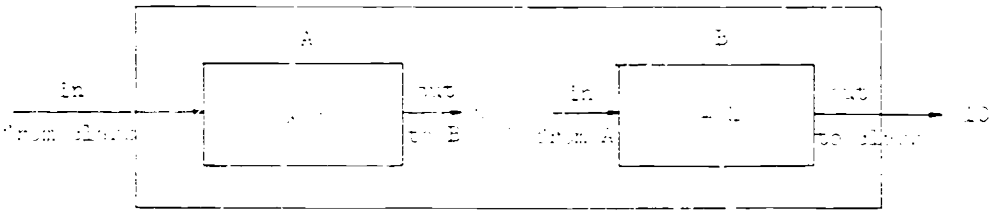
2. A member of the class selects a number, the input, which is communicated to the machine.

3. The machine announces the number, the output, into which his rule converts the input. Thus, in the example cited, if 5 is the input he announces "15" as the output.

4. The "audience," i.e., the class, now try to identify the operation of the machine. In the example cited above this is very simple and if the machine is an ordinary multiplier, one input will always suffice. In slightly more complex arrangements the situation is [illegible].

[illegible] of the simple machine described in 1. above [illegible] "[illegible]" [illegible]. One pupil A is an unknown [illegible] unknown adder, i.e., he adds a fixed number [illegible]. The input number is [illegible].

For example:



Thus in this case the compound hook-up converts a [illegible] input to a 10 output. Similarly, if 1 is fed into the machine [illegible] comes out and if 5 is fed into the machine [illegible] comes out.

The problem [illegible] for the analysts is to identify what each component, A and B, is doing. A number of third grade students are capable of discovering quickly that the hook-up can be identified by testing it with any input. [illegible] will [illegible] almost immediately that a zero input gives away the adder, B.

SUGGESTED GAMES

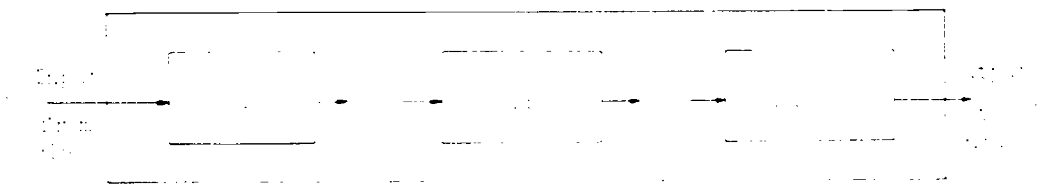
SINGLE COMPONENT GAMES - Games 1 through 6

Function Game 1

In introducing the game the teacher can play the role of the machine. [illegible] is a simple "add [illegible]" game. The teacher announces that she is a machine which adds the same amount to each number that is given to her. The children are asked to suggest inputs. The teacher then silently operates on the numbers and reveals the results in the form of outputs. It would be advisable to list the pairs of inputs and outputs on the board as follows:

Input		Output
[illegible]	→	[illegible]
[illegible]	→	[illegible]
[illegible]	→	[illegible]

From the [illegible] it [illegible] clear that the teacher is adding [illegible] to each number. [illegible] need only the input to [illegible]. After the teacher has played the game several times, a student would be asked to play the role of the machine. Then the student [illegible] in front of the room and [illegible] another student who will record the inputs and outputs on the board. [illegible] in concert, pairs of



For example:

[illegible]

[illegible]

[illegible] and what adder [illegible] For example:

[illegible]	[illegible]	add [illegible]	multiply by 2
[illegible]	[illegible]	Input	Output
			10

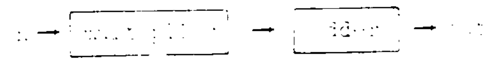
[illegible] of the first machine. [illegible] is immaterial.

[illegible]

[illegible] and division page 20 [illegible] a facility for inventing [illegible] of the two [illegible]

[illegible] children try to discover [illegible]

For example:



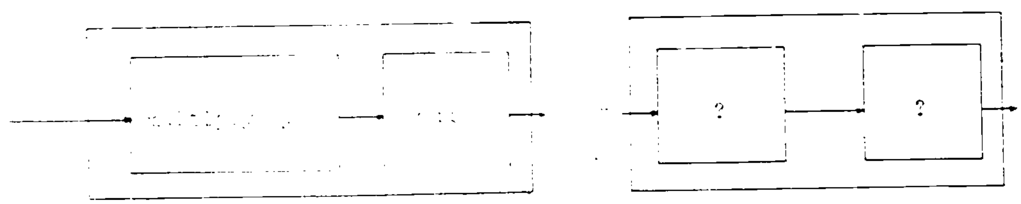
Input	Output
[illegible]	[illegible]

... children should [illegible] and then [illegible].

The [illegible] "two components" which will take the [illegible] output [illegible] from the [illegible] machine and [illegible] them back into the original inputs. [illegible] of what we are after, place two [illegible] the two students who are acting as the original multiplier-adder machine. [illegible] the diagram indicated below on the board.

For example [illegible] the machine [illegible] yielded output 8.

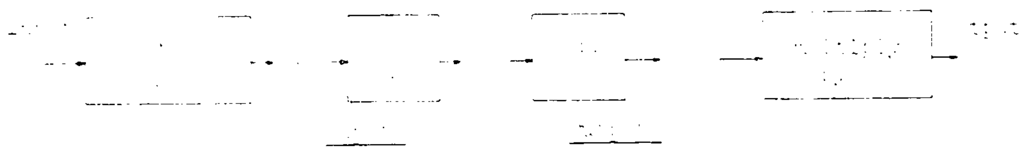
We want [illegible] input into the machine, [illegible] comes out. [illegible] In diagram form:

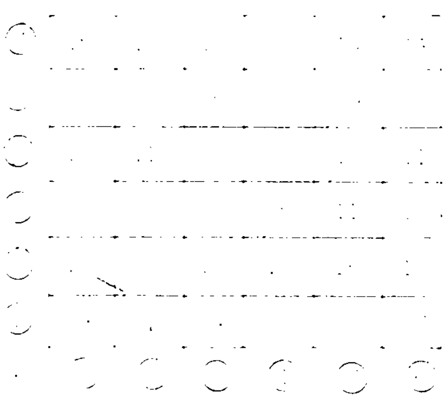


Draw this on the board with explanation.

Input	Output	Input	Output
			0
[illegible]		[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]

Children [illegible] type of "[illegible]" operation [illegible]. In [illegible] would [illegible]. This is [illegible]. The children find it more dynamic and meaningful.





In the [illegible] initial work with [illegible] multiplication tables, the children might not have noticed that the ratio of the numbers in column 1 to the numbers in column [illegible] was [illegible]. Also the ratio of the numbers in row 1 to the numbers in row [illegible] is constant. [illegible]

[illegible] from the numbers immediately below in row [illegible]. Thus from the [illegible]

[illegible] For example: [illegible] multiplication table.

[illegible]

[illegible] multiplication table [illegible]

[illegible] common factor [illegible] or more numbers.

[illegible] or more numbers.

[illegible]

INTRODUCTION. It is a common occurrence for teachers to have students who are fascinated with mathematical puzzles but who show little interest in the usual textbook exercises. Furthermore, even those students who faithfully do their problem assignments are rarely stimulated and challenged by the problems themselves. These observations suggest that the effectiveness of problem assignments would be very greatly increased if the usual drill-type exercises could be systematically replaced by puzzle-type problems and games which make use of the same mathematical skills. However, the construction of such problems and games is not easy, and most of the development has been directed toward recreational uses. Nevertheless, many educators feel that this material has a great potential for educational use, and have urged that a concentrated effort be made to develop mathematical games and puzzle-problems for this purpose. A proposal along these lines was made to the School Mathematics Study Group which has broad interests in mathematics curriculum development. After obtaining approval from the National Science Foundation, SMSG agreed to support a two-week conference to explore the possibilities for the development of mathematical games and puzzle-problems which would be appropriate for educational use. The basic aim of the conference would be to generate ideas. There would be no construction of mathematical gadgets or preparation of text material.

THE CONFERENCE. The conference was held at Stanford University during the period [illegible] - June [illegible], [illegible] with the following mathematicians participating:

Professor R. P. Dilworth, California Institute of Technology
Dr. Walter Jacobs, Department of Defense
Professor William Lister, The State University of New York
Dr. Frank Sinden, Bell Telephone Laboratories
Professor C. J. Titus, University of Michigan
Mr. Herbert Wills, University of Illinois
Mr. Robert Wirtz, Carmel, California

All of the participants have extensive interests in the area of mathematical recreation and most of them have been involved in a variety of curriculum development activities.

The formal sessions of the Conference were essentially brainstorming sessions in which a variety of ideas for puzzle-problems and games were proposed by the participants for consideration by the group as a whole. The items which survived after detailed discussion and criticism were assigned to conference participants to be written up for the final report. On the average one such session was held each day. The remainder of the time was devoted to an extensive examination of the literature on mathematical recreations; to modifying, extending, and translating known recreational items in order to make them suitable for educational use, and to preparing reports on the results of these endeavors. The principal output of the conference is the collection of working papers which is attached as an appendix to this report.

<u>CONCLUSIONS AND RECOMMENDATIONS</u>. First of all, it is clear that the development of an entirely new class of puzzle-problems or games is a very difficult undertaking. The abilities required are not unlike those required in developing an entirely new line of mathematical research. The essential

feature which distinguishes good puzzle-problems from ordinary textbook problems is the sense of elegance, novelty, and in some cases, suprise which they generate. Likewise, an effective game must present a natural, attractive challenge to the players. But it is exactly these features which characterize first-class mathematical research. Hence, it is to be expected that outstanding contributions to the art of puzzle-problem construction will be rare. However, it also seems to be the case that the discovery of a new idea will frequently lead to a wide variety of puzzles and games. The many puzzles and games which have developed from S. Golomb's invention of polyominoes illustrates this point. During the relatively short period of the conference, some novel ideas for puzzle-problems and games were produced. However, it will take further development and experimentation to determine how effective and how broadly applicable the ideas are. In any case, it was the unanimous feeling of the participants that a joint effort such as this is an effective way to stimulate ideas in this area. They further recommended that similar conferences of this type be held from time to time to encourage creative activity in this area and to make use of the ideas and talents of other people. It was felt that a two-week period was about the right length of time since after a fortnight of intensive effort the wells of creativity appear to temporarily run dry.

Many of the working papers do not present entirely new ideas for puzzle-problems and games but rather give ideas for the development of known material in order to increase its potential for educational use. This is clearly a very important activity for such a conference and in many cases much imagination and resourcefulness [illegible].

No attempt was made to present the material in polished form. The working papers serve merely to get the ideas down in a preliminary form suitable for later experimentation and modification for use in curricular material.

All members of the Conference are agreed that the next stage in the development of this material should be experimental testing on an item by item basis. This will require a good teacher as well as a mathematician. Furthermore, simply turning the material over to a good teacher will not be enough since the objectives which the inventor had in mind may very well be lost in the process of adapting the item to classroom use. Perhaps the ideal arrangement would be to have the experimentation done by a team of two -- a good teacher and a mathematician. If the mathematician involved could be the man who prepared the item, so much the better. Under any circumstances it is the consensus of the Conference that until there has been such experimentation, the material should not be widely circulated. In particular, it should not be made available to textbook writers in its present form.

After experimental testing, the next stage should be the development of some sample text material incorporating the tested items. This could be a responsibility of a writing team. On the other hand, it is the feeling of the Conference participants that this should be approached in an imaginative way. For example, rather than trying to fit puzzle-problems and game activity into the traditional text format, it might be much more fruitful to design the text material around the problems and game activities, so that the learning of mathematical facts and the development of mathematical skills is a natural by-product of the interest [illegible] activities.

Finally, the participants wish to express their appreciation for the cheerful and hearty cooperation of the SMSG staff.

R. P. Dilworth, Chairman

APPENDIX

The working papers produced by the Conference participants are collected in this appendix. It must be emphasized that these papers were prepared in order that there would be a permanent record of the ideas generated during the Conference. No attempt was made to polish the presentation nor was any effort made to be complete and detailed in the discussion of the item. Rather the motivation was to get the basic ideas down in writing as quickly as possible and to get on to the generation of more ideas. Furthermore, the form of the presentation was that which was most convenient for the particular participant involved, and no effort was made to get uniformity.

It should be noted that the working papers vary from short accounts of very particular items to broad descriptions of whole classes of games and puzzles. The grade level extends over the primary and secondary spectrum. In many cases, the activity could be profitable for primary students while an insightful analysis could only be carried out by advanced secondary students. For example, Nim can be played by primary students, but they are hardly able to analyse the game. Finally, it should be mentioned that the items fall into three categories -- classroom activities, individual puzzle-problems, and games. Some of the items naturally fall into more than one category, but a rough distribution of the items is as follows:

Classroom presentation: 1, 2, 3, 6, 11, 20, 21, 24
Puzzle-problems: 13, 14, 15, 17, 19, 21, 22, 23, 24
Games: 1, 4, 5, 6, 7, 8, 9, 12, 16, 18, 25, 26, 27

A rough classification of the items according to grade level is as follows:

Grades 1, 2, 3, 4: 1, 2, 3, 5, 7, 8, 9, 11, 12, 13, 16, 17, 18, 20, 26

Grades 5, 6, 7, 8: 1, 2, 3, 4, 6, 10, 11, 13, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27

Grades 9, 10, 11, 12: [illegible] 4, 15, 17, 19, 21, 22, 24, 27.

ARITHMETIC FUNCTION GAMES

A great variety of classroom games are suggested by the small sample listed below. They should, incidentally, provide practice in basic arithmetic skills, but their focus is primarily on (1) properties of numbers with respect to operations, (2) the function idea, and (3) the problem of inferring a [illegible] from experimental data.

I. Linear functions (grades [illegible])

A student (the "machine") plays a linear function, e.g. multiply by [illegible] then add [illegible]. The class feeds inputs to the machine which then announces the output. The problem is to identify the machine's operation.

II. Composites of functions (grades [illegible])

The teacher assigns to one student a multiplier (say [illegible]) and to another an addend (say [illegible]). The class does not know which is a multiplier nor what numbers have been assigned. The class hooks up the students in series and feeds inputs to A. He whispers his output to B who operates on it and then gives the composite machine's output to the class.

IN ⟶ [A] ⟶ [B] ⟶ OUT

The problem is to infer what each component is doing. The idea of [illegible] of machines and of standard machines, defined as those in which the multiplier is in A, will come up naturally here.

III. Remainder functions (grades [illegible])

The teacher assigns a student only a divisor (say [illegible]). For every positive integer the student-machine computes the remainder after division by [illegible] and announces it to the class. The problem is to identify the divisor.

A variation is to have the class (each with a number less than say 100) [illegible] remainder machines until it is identified.

IV. A [illegible] ([illegible])

[illegible] to the [illegible]. For example, [illegible] classes [illegible] them in the [illegible]. The problem is [illegible] find [illegible] of this [illegible] with rational inputs where the [illegible], proportional [illegible] normal [illegible].

W. [illegible]

[illegible]

[illegible]

[illegible]

[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]

[illegible]

The [illegible] is shown [illegible]. The [illegible] and [illegible] alternately [illegible] (with [illegible] replacement) [illegible] each attempting to make [illegible] using one or more [illegible].

[illegible]	[illegible]		[illegible]		[illegible]		
[illegible]		[illegible]		[illegible]		[illegible]	etc.

This game will probably end in a win for neither side, but at first the [illegible] will frequently win or first player will never lose. The strategy can [illegible] by drawing an associated magic square.

With this magic square no sums of [illegible] entries can win and with [illegible] entries only [illegible] row, column and diagonal sums win, hence the game is equivalent to tic-tac-toe. The question arises whether all magic squares yield games equivalent to tic-tac-toe, [illegible] whether in particular the sum of [illegible] entries or the sum [illegible] not form a row, column or diagonal can win.

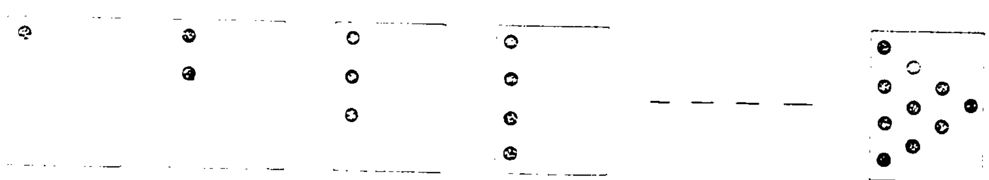
Presumably, [illegible] students will attempt to find a systematic [illegible] magic squares.

[illegible] Wirtz

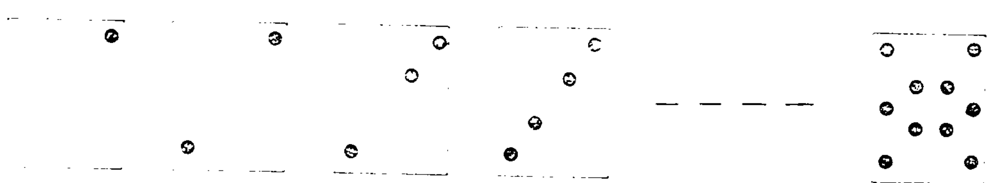
[illegible]

[illegible], or [illegible] number [illegible] from [illegible]. A sample of these [illegible] of poor suitability for [illegible] room [illegible] small group activities which [illegible] school. [illegible] provide natural context in which basic [illegible] introduction [illegible].

[illegible] a [illegible] special [illegible] consist of [illegible] configurations. For example the [illegible]:



[illegible]:



[illegible] numeral [illegible] four, [illegible] further [illegible] only with numerals in the various colors.

[illegible]

[illegible] all of the [illegible] described above may be used in this [illegible]. [illegible] objective in the earliest [illegible] is to develop [illegible] with respect to number. Numerals are not [illegible]. [illegible] marked with numerals should [illegible] the number-numeral associations.

[illegible] player. [illegible] players.
[illegible] the player [illegible] [illegible].

[illegible]

[illegible] player one [illegible] in front of the player. [illegible] row in front of the [illegible] number with one [illegible]. If he proceeds properly [illegible] in the row in front of him. [illegible] the literal [illegible].

[illegible]

[illegible]. [illegible] players.

[illegible] the player holding the [illegible]. [illegible] player [illegible] one [illegible] each row [illegible] row [illegible] in proper order. [illegible] player [illegible] play must [illegible] will promote his own [illegible] opponent. The first player to play [illegible] number of unplayed [illegible] player the number of cards he has left in [illegible].

[illegible]

[illegible] pr[illegible] to the simplifi[illegible]. [illegible], [illegible]. [illegible] of [illegible] play [illegible] in [illegible] variation.

[illegible]

[illegible] all, more compl[illegible] many. [illegible] provide [illegible] ordering, [illegible] family [illegible].

Al[illegible] the rules of conventional crib[illegible], learned in play, [illegible] will not be described here. A few simple [illegible] to the simplified board. The [illegible] replaced by 10; [illegible] the upper [illegible] replaced by 1. In the simplest version the "crib" can be eliminated [illegible] cards [illegible] to each player. The game is best [illegible] with parallel rows [illegible] which each player moves [illegible] points.

W. [illegible]
H. [illegible]

[illegible]

A. [illegible]

1. [illegible] is probably most important for [illegible] the way they express [illegible] by [illegible] single [illegible].

2. This [illegible] basis [illegible] of classroom activities [illegible] structure and teamwork, and at the [illegible] the class.

3. [illegible] provide [illegible] preparation for the [illegible] computer programming; this is currently being taught to [illegible], and it is [illegible] that it will be taught [illegible] pupils.

B. [illegible]

1. [illegible] "Binary [illegible]", which [illegible] five students to [illegible], illustrates some of these [illegible].

2. [illegible] "flip-flop", and [illegible] his state (from arm [illegible] arm [illegible]) [illegible] an effective signal. [illegible] indicated by the [illegible] from the preceding [illegible].

3. [illegible] participant has a fairly simple [illegible] his activities, [illegible] practice and [illegible] avoid mistakes. Since [illegible] only when every [illegible] his part correctly, [illegible] in teamwork.

4. [illegible] expression [illegible] concept of [illegible] rules [illegible] correct [illegible] grasp of number bases.

[illegible]: [illegible] the initial [illegible] number [illegible] [illegible], [illegible] number [illegible] [illegible] [illegible] number [illegible]:

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible] ---	down	down	down	down	down	[illegible]
[illegible] ---	down	down	down	down	up	[illegible]
[illegible] --	down	down	down	up	down	[illegible]

[illegible]

[illegible] number [illegible] "[illegible]" [illegible] represented [illegible] at the end of any [illegible] number represented [illegible] of the initial number and the number of [illegible]. When the class under- [illegible] predict what the result will be in [illegible], starting with [illegible].

[illegible]: [illegible] down from an initial number, the rule is that [illegible] position [illegible] "[illegible]" [illegible] in front raises his arm, and simply [illegible] in front drops his arm. It will take [illegible] bright [illegible] the [illegible] in the [illegible].

[illegible] arms, the same type of [illegible] can demonstrate [illegible] represented by

[illegible] - both arms down

[illegible] - [illegible] arm up

[illegible] - both arms up

Example [illegible]. [illegible]

[illegible]: To provide more [illegible] similarly the operation of [illegible].

Participants: Two or more teams, each of five players, who take parts respectively of [illegible] Position, Threes Position, Nines Position, Twenty-sevens Position, Eighty-ones Position.

Description: Each team performs as a base [illegible] adding machine, and attempts to add a series of six [illegible] numbers as rapidly as possible. The teams may compete in pairs, or race against time. They can be given freedom to choose among different procedures, ranging from deliberate and reliable to rapid and precarious methods [illegible] because of the likelihood of a mistake. A class discussion of possible methods will help to bring out the commutativity of addition at the [illegible]-operation level.

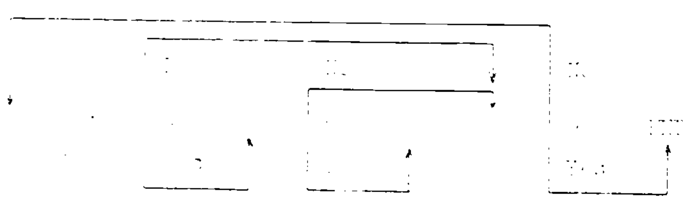
Procedure: The method of operation follows principles similar to those [illegible] in connection with the Binary Counter Game. Each Position exhibits a digit in base [illegible] by holding up a corresponding number of arms, and adds another digit by counting forward from the current position. For example, if one arm is up, corresponding to the digit "1," and the digit "[illegible]" is to be added, the Position raises the second arm and silently counts "One", then drops both arms and silently counts "Two". Having reached the digit to be added, he is done. The next higher position must have been watching, and must add "1" as a carry at the point where the other has dropped both arms.

In detail, the procedure for adding a base 3 number to one already represented in the arm positions of the team is as follows: the step-by-step method [illegible]:

1. Ones Position adds the units digit of the new number, which is written on the blackboard. He signals that he is done by making a quarter turn to face Threes Position.

2. Threes Position makes a quarter turn to face the blackboard, and adds "1" [illegible] there has been a carry signaled by Ones dropping both arms. He continues [illegible] the threes digit of the number on the blackboard, and when he is done [illegible] turns to face Nines.

3. This continues until Eighty-ones has finished his addition. At this point he calls out "Ready"; [illegible] turns to face the board; all others turn to face him; and the team is set to start adding the next number.



[illegible] as follows. These are the different possibilities, [illegible] which proceeds as in the illustration below :

STEP	PARTICIPANT	ACTION
	Counter	Holds 1, and reads "1" to Number.
	Number	Reads to Units the digit on line named by Counter.
	Units	Adds to number held the digit named by Number.
	Units	Tells "NO" if sum is less than 10, "YES" if sum is greater than or equal to 10. Retains units digit.
	Tens	Adds 1 to number held.
	Tens	Tells "NO" if sum is less than 10, "YES" if sum equals 10. Retains units digit of sum.
	Overflow	Displays 1.
	Counter	Adds 1 to number held.
	Counter	Tells "NO" if sum is less than N, "YES" if sum equals N the total number of digits to be added.

When [illegible] is reached, the final total is given as the pair of digits held by Tens and Units, provided that Overflow still shows 0. If he displays 1, the correct answer differs from that held by some multiple of 100.

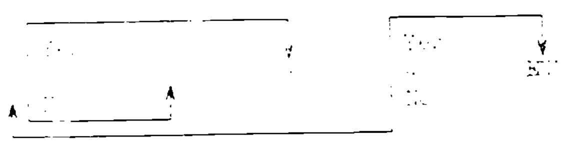
Each participant may hold a card showing the step numbers at which he takes action, and the nature of the corresponding action. Even in this case, it will [illegible] the teacher [illegible] as the number of the current step is [illegible].

The [illegible] shows the sequence of calls and numbers held for the first [illegible]. It is easy to verify that this sequence corresponds [illegible] Number's list begins [illegible] ...

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	OVERFLOW
Start			[illegible]	[illegible]	0
[illegible]	[illegible]		[illegible]	[illegible]	0
[illegible]	[illegible]	Reads "[illegible]"	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]		[illegible]	[illegible]	[illegible]
[illegible]	[illegible]		[illegible]	"No"	[illegible]
	[illegible]		[illegible]	[illegible]	[illegible]
[illegible]	[illegible]		[illegible]	[illegible]	[illegible]
		Reads "[illegible]"			[illegible]
[illegible]			[illegible]	11	[illegible]
[illegible]			[illegible]	"Yes"	[illegible]
[illegible]			[illegible]	[illegible]	[illegible]
[illegible]			"No"	[illegible]	[illegible]
	[illegible]		[illegible]	[illegible]	[illegible]
[illegible]	[illegible]		[illegible]	[illegible]	[illegible]
		Reads "[illegible]"	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]			[illegible]	[illegible]
[illegible]	[illegible]		[illegible]	"Yes"	[illegible]
	[illegible]			[illegible]	0
[illegible]	[illegible]		"No"	[illegible]	[illegible]
	[illegible]			[illegible]	[illegible]
[illegible]	[illegible]				

[illegible] the next cycle [illegible] corres- [illegible] Tens and Units.

[illegible] provide the capability [illegible] complementing the [illegible]



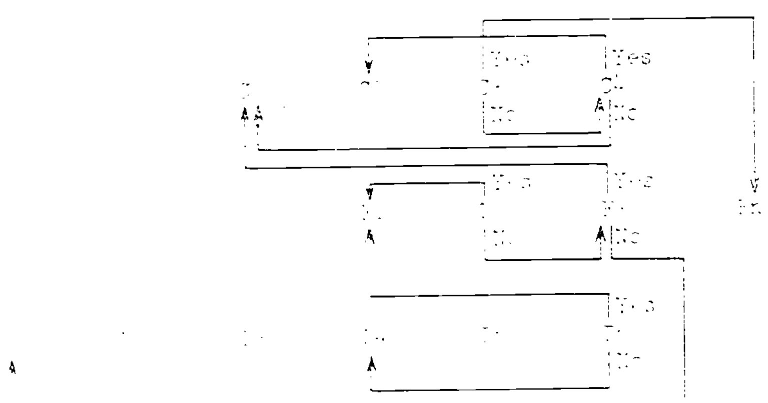
Example _. [illegible] Primes Less Than N

Purpose: [illegible] prime numbers, and [illegible] of subroutines.

Participants: Seven students, called, respectively, [illegible], Find, Copy, [illegible], [illegible], Next, and Number.

Description: The program [illegible] alternating two subroutines until the prime [illegible] $\sqrt{N}$ [illegible] (N [illegible] for the example). Then the third [illegible] remaining primes not greater than N. The participants [illegible], [illegible] and [illegible] control the sequence of steps in their respective subroutines, and the other four students perform specific functions. The program [illegible] with the numbers [illegible] to N written on the blackboard, and the Delete [illegible] the latest prime determined to be removed from [illegible], together with its multiples starting with the square of that prime. [illegible] works properly. The Find [illegible] the next larger prime to be located, and if this prime [illegible] $\sqrt{N}$, it returns control to Delete; otherwise control passes to [illegible], [illegible] the remaining primes to be listed directly.

Sequencing: The [illegible] as the first prime, and this [illegible] Next. Then [illegible] starts his subroutine. The sequencing of the [illegible] following diagram, in which the steps of each [illegible] separate line:



These steps are described in the following table:

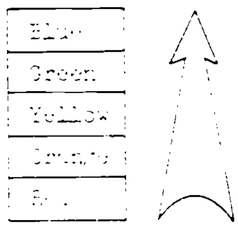
[illegible]	PARTICIPANT	ACTION
[illegible]	List	Enters the number held by Next on the list of primes.
[illegible]	Erase	Erases from the table the number held by Next.
[illegible]	Number	Calculates the square of the number held by Next.
[illegible]	Erase	Erases from the table the number held by Number.
[illegible]	Number	Adds to his number the number held by Next.
[illegible]	Number	Calls "Yes" if his sum is greater than "100," otherwise calls "No."
[illegible]	Next	Adds 1 to the number he holds.
[illegible]	Next	Calls "Yes" if the number he now holds has been erased, otherwise "No."
[illegible]	Next	Calls "Yes" if the number he holds is greater than 10, otherwise "No."
[illegible]	List	Enters the number held by Next on the list of primes.
[illegible]	Next	Adds [illegible] to the number he holds.
[illegible]	Next	Calls "Yes" if the number exceeds 100, otherwise "No."
[illegible]	Next	Calls "Yes" if the number has been erased, otherwise "No."

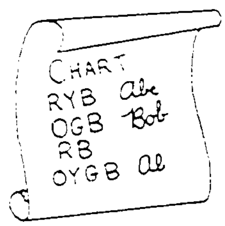
Find initiates his subroutine when "Yes" is called on Step [illegible]. Delete performs his when Next calls "No" on Step [illegible]; if instead he calls "Yes," Copy begins his subroutine. At this point, all primes $\leq \sqrt{100}$ have been listed, and the numbers in the table that have not been erased are all primes. (The class should find the reason for this rule.)

It will often happen that Erase is called on to remove from the table a number that has already been erased. (When will this occur?) In this case he does nothing. It can be pointed out in discussion that erasing a blank leaves a blank, and that making use of this fact simplifies the program; otherwise the table would have to be examined before deciding to erase. Also, the need for a subroutine Copy should be discussed for its insight into the storage of information: since the numbers that remain when this subroutine begins are irregularly scattered, some method of scanning them is required. Visual scanning is so automatic that its occurrence tends to be overlooked.

[illegible]

JUMPING GAME





Five (the number is optional) boards 3 feet by 10 inches are joined together so that they will not separate when children jump upon them. The object of the game is to jump onto the blue (last) board successfully. A player is not successful should he land on a line or should he copy a trip made by a fellow player. A trip may consist of any number of jumps of any length as long as the jumps proceed toward the blue board. That is a player may not jump back toward the [illegible]. The trips always start on the floor and end up on the blue board. Also, [illegible] must commit himself to a specific trip, before he starts his jumps, [illegible] recording his proposed trip on the chart. If he completes his trip successfully he enters his name along side of his proposed trip. An unsuccessful [illegible] try his trip again on his next turn unless someone else has taken it successfully. If either of these situations arise he may try a new trip if he can think of one that has not been used up. For example, suppose [illegible] himself to jumping directly on the yellow and then on the blue. [illegible] 'YB' on the chart. Thus, all of the trips that have been used up [illegible] the chart.

The player with the most successes wins the game. This game should be limited to a maximum of [illegible] players. The winning player starts last for the next game. Boards may be added or taken away depending upon the ability of the players. This game is useful for developing systems for exhausting all possibilities as well as giving very young students experience in extending their [illegible] one, two, three, and so forth by such activities as:

If you start on the orange board and take a jump of two, where will you land?

If you start [illegible] on the red board and landed on the green board, how long was your jump?

If you landed on the blue board by taking a jump of [illegible], where did you start?

How many jumps of [illegible] would you need to take to get you as far as a jump of [illegible]?

If you took a jump of [illegible] and then a jump of [illegible], how far would your friend need to jump to get to where you are in one jump?

etc.

[illegible] activity [illegible] motivates the interesting problem:
How many trips are possible using, say, [illegible] boards?

This problem lends itself to several approaches depending upon the level of the student to whom it is presented. To illustrate a few of the possibilities we might consider approaches appropriate for elementary students, junior-high students, and high school students, respectively.

- Suppose I had just one board. How many trips are open to me? For two boards? Three? Let's list our results:

Boards	1	[illegible]	[illegible]	4	[illegible]	...
Trips	1	[illegible]	[illegible]	8	?	...

Suppose I have played the game with [illegible] boards and find that I have [illegible] possible trips. Now I add one more board. Consider each of the possible trips in the [illegible]-board game. Each of these trips ended by jumping onto the blue board, say. But now instead of jumping on the blue board, I have two choices. I can jump on the blue board or directly onto the new last board. Hence, I have twice as many possible trips as before. This gives me a recursion formula for determining the number of possible trips since I know that with 1 board I have only 1 trip.

$$\boxed{\begin{array}{c} p(1) = 1 \\ \text{and} \\ \forall k \; p(k+1) = 2 \cdot p(k) \end{array}}$$

c) For those students who have discovered and verified that the number of subsets of an n element set is 2^n, another solution is available. Since each trip ends on the blue board, the set of boards jumped upon before landing upon the blue board determines a trip. For example, what trip corresponds to the empty set of boards? Hence, the problem simply boils down to, "How many different sets of non-blue boards are there?" This of course is the number of subsets of a set consisting of k-1 elements where k is the total number of boards including the blue board.

H. W[illegible]

SEQUENCE MAGIC

Many teachers are aware of the fact that children like to puzzle out the "next" term of a sequence after having been given several preceding ones. In some cases, however, the students are left with the impressions listed below.

1. There is only one sequence with the terms given.
2. There must always be a next term.
3. Sequences must be determined by some "pleasant" rule.

It is unfortunate that such a pleasant pastime results in some unhealthy thoughts. In order to retain these merits and at the same time eliminate some harmful side effects, the following suggestions are made.

Present a sequence of terms like:

$$[illegible], 7, 10$$

and ask students for the next term of a sequence which has the beginning shown above. After they give their next term they are to give some kind of rule for generating the whole sequence. This approach promotes creativity and graphically illustrates that asking for the next term of an unspecified sequence is inappropriate. This activity could be started with less given terms and gradually worked up until the game seems quite challenging.

For instance one might start by giving merely two as the first term. Then students might be given:

$$2, 4$$

then

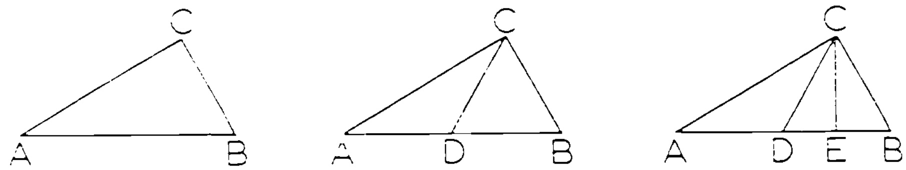
$$[illegible], 4, [illegible]$$

Sequences which start off:

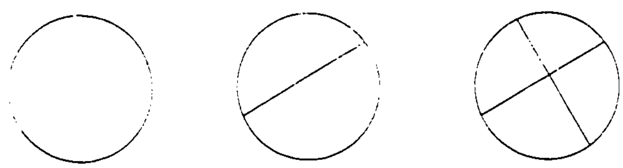
$$[illegible], 4, [illegible]$$

The possibility of generating other than the sequences which [illegible] expected [illegible] (those which start in this way but vary from the powers-of-two sequence) [illegible]. Here are a few examples:

Consider drawing line segments from the vertex of a triangle to its base. With no extra line segments drawn you have only one triangle pictured. With one extra line segment drawn you have two non-overlapping triangles pictured. Now, someone who is really fast on the [illegible] might conjecture that adding [illegible] with two line segments we will have four non-overlapping triangles. This is not the case as is illustrated below.



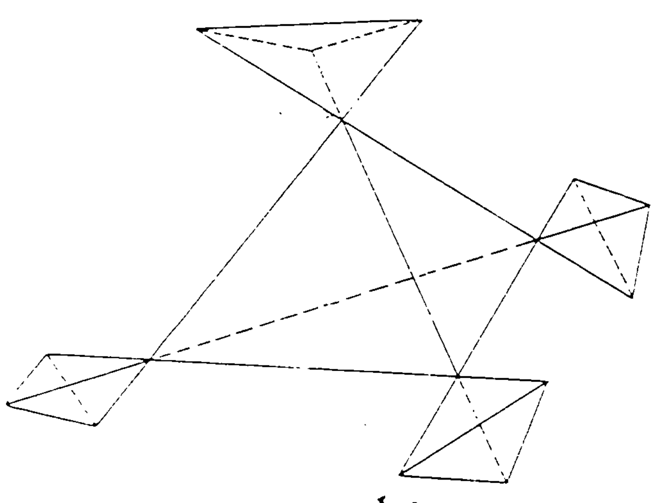
[illegible] geometric situation which generates a sequence which starts out 1, 2, 4. [illegible] chords in a circle in such a way that the maximum number of regions are formed:



0 chords 1 chord 2 chords

[illegible] chords we get a maximum of [illegible] regions.

[illegible] geometric situation yielding a sequence which starts out 1, 2, 4, 8. The question is: how many 3-dimensional regions are formed by [illegible] planes. Zero planes, one plane, two planes, [illegible] regions respectively. He furnishes a [illegible] drawing for the case of 4 planes in space:



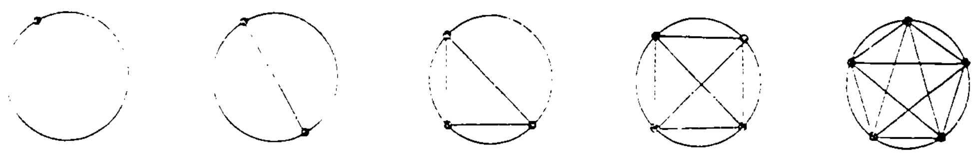
[illegible]

[illegible]

[illegible]

[illegible]

[illegible]



[illegible]	[illegible]
[illegible]	[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] possibility. If the numbers listed are [illegible]:

[illegible]

[illegible] also [illegible] constructing sequences from recursive rules which yield [illegible]. There are several examples of these in the CMSP-ITEMA "Gold Mine".

[illegible] sequences may be used to familiarize students with certain patterns which come up again and again in mathematics. For instance we have [illegible] number of situations where the powers-of-two pattern comes up [illegible] on the Jumping Game and the Trains of Rods. A sequence [illegible] appropriately [illegible] into classroom activity when it precedes a [illegible] the same or related pattern. They may also be used immediately [illegible] discovery of a pattern so as to give students more experience with [illegible] important for kids since they could not easily discover a pattern with which they were unfamiliar. To make this point clearer, consider a class [illegible] in squaring numbers. When these students [illegible] and [illegible] they immediately recognize them [illegible] another class without this experience. Now face them [illegible] with questions [illegible]:

What is the sum of the first [illegible] odd numbers?
What is the sum of the first [illegible] odd numbers?
[illegible] odd numbers? [illegible]

[illegible] class with the experience described above [illegible] question such as the [illegible] successful.

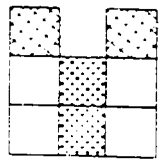
[illegible] first [illegible] numbers?

[illegible] next term.

[illegible]

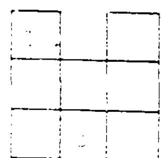
[illegible] "Y"

[illegible]

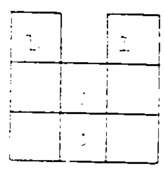


[illegible] "Y" [illegible] arrangement of the shaded [illegible] [illegible] "Y" [illegible] must be entered in the remaining cell [illegible] Similarly, when a numeral is written in either of the cells [illegible] of this numeral must be placed in the other cell.

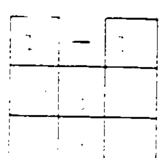
[illegible]

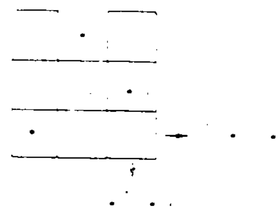


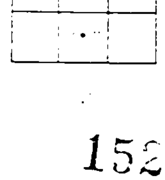
[illegible] in this particular [illegible]

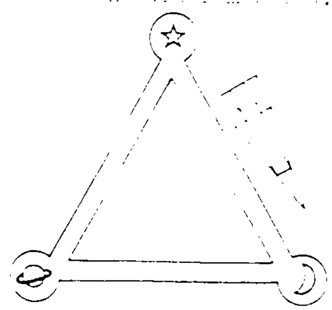


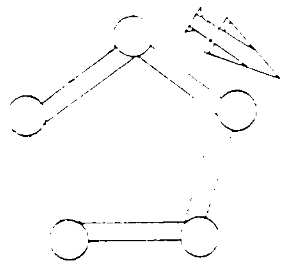
[illegible]

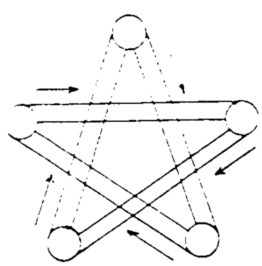


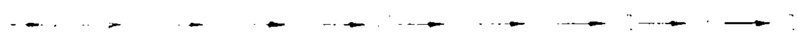








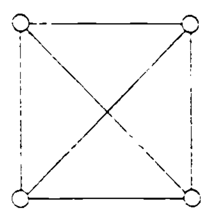




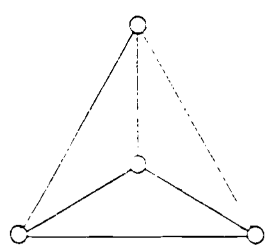
[illegible]

[illegible] ... paper and pencil exercise [illegible]

[illegible] ... pair is connected. [illegible]

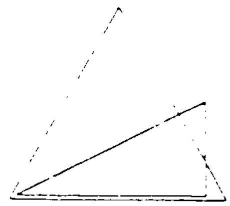


[illegible]

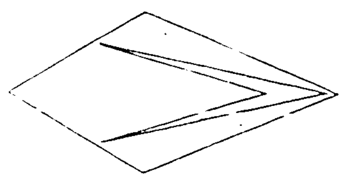


[illegible]

[illegible] example:



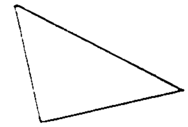
[illegible] is a [illegible] quadrilateral [illegible] quadrilateral [illegible]

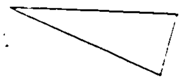


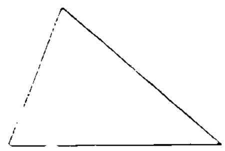
[illegible]

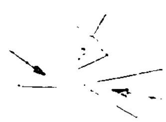
[illegible] proposed by a Russian mathematician [illegible] center of [illegible] and [illegible] main diagonals, in what [illegible] sufficiently vivid mental image [illegible] can often [illegible] arrive at this by [illegible] faces have a common vertex [illegible] have a common vertex at the [illegible] visualization. [illegible] by symmetry, [illegible] by symmetry, the line [illegible] a proof of [illegible]

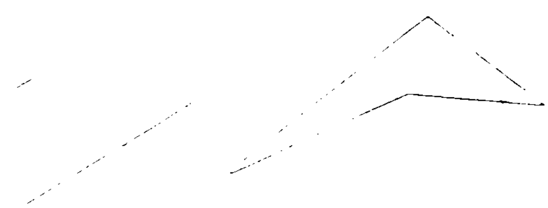
[illegible]

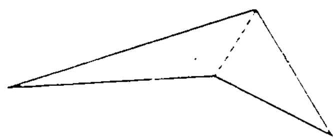


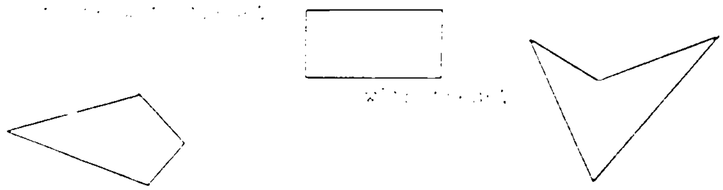


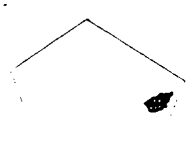


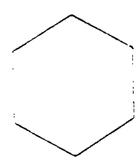


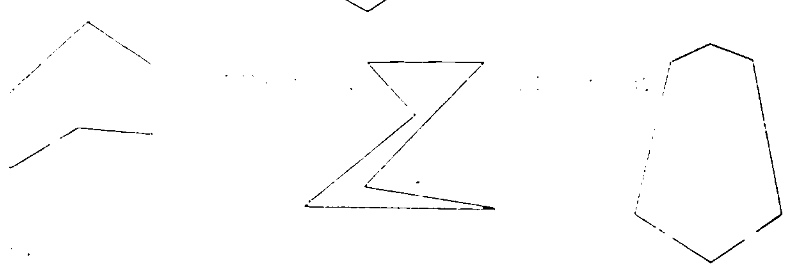


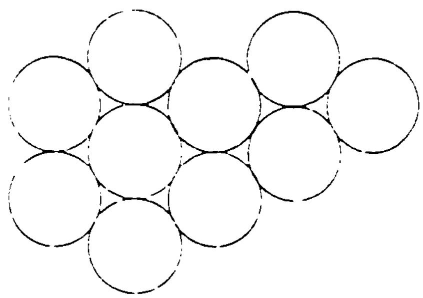


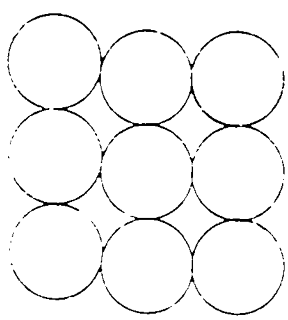


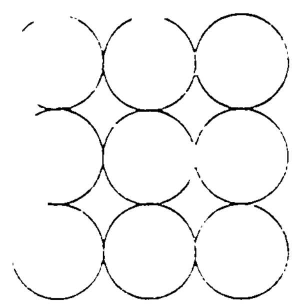




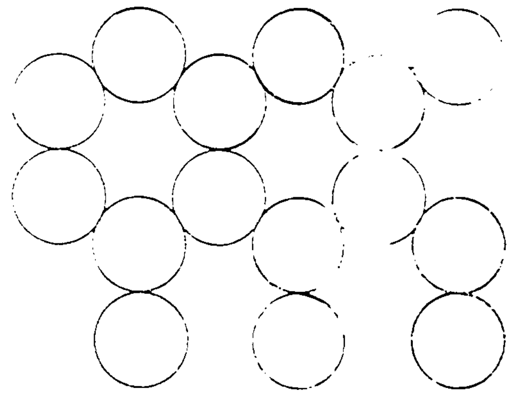








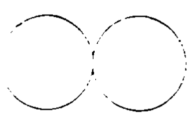
[illegible]

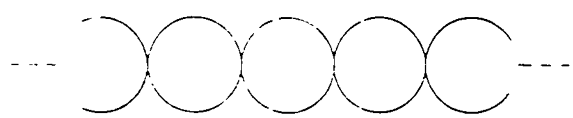


[illegible] each sphere surrounded by [illegible]

[illegible] each sphere surrounded by [illegible]

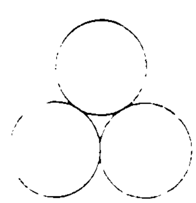
[illegible] -packing [illegible] -packing





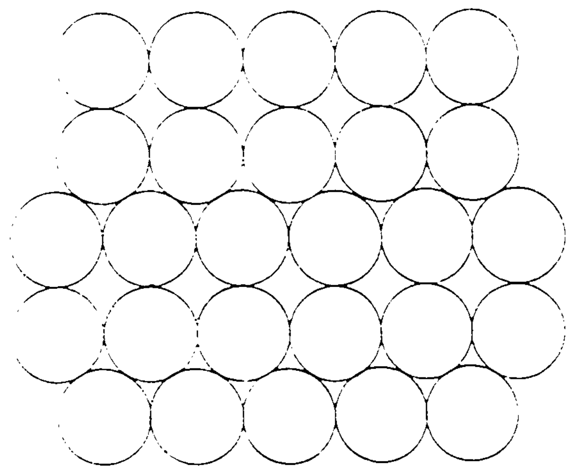
[illegible] packing is infinite in [illegible] is infinite in all directions.

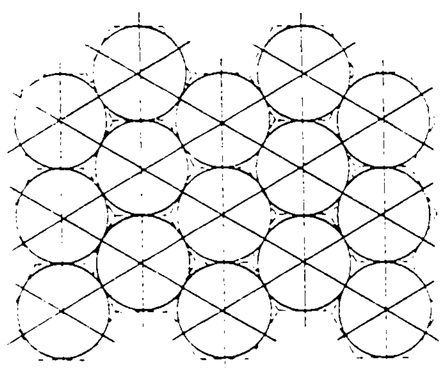
[illegible] -packing



[illegible]

[illegible]

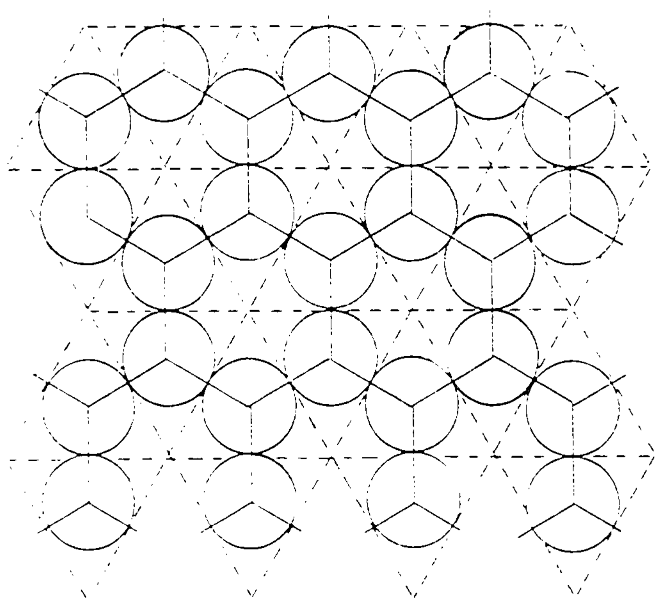




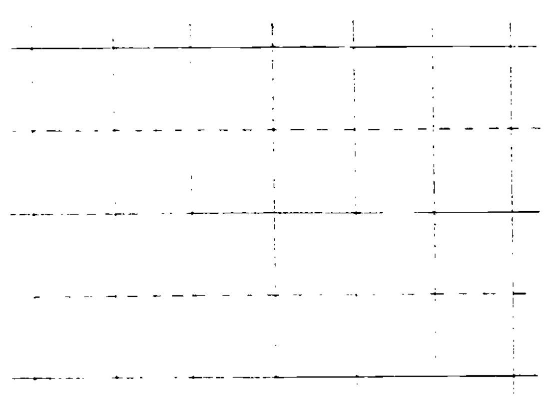
Solid lines join centers of circles; dashed lines join centers of spheres. [illegible]

Cells formed by solid lines have ___ sides.

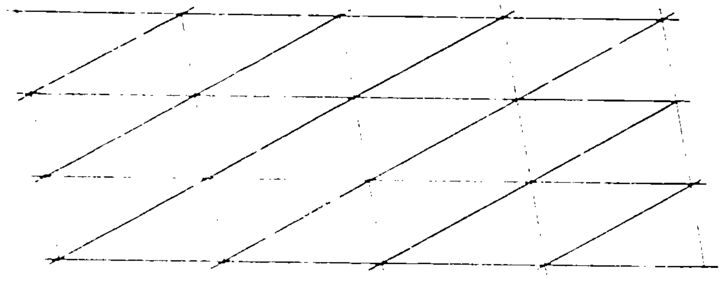
Cells formed by dashed lines have ___ sides.

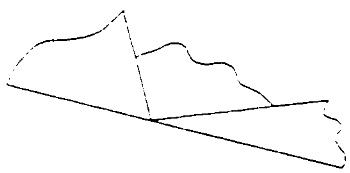


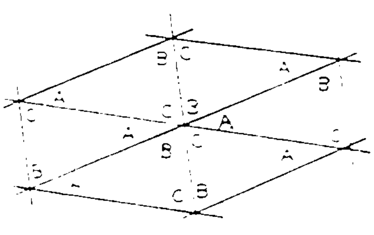
[illegible] dashed lines are the [illegible]
[illegible] The [illegible] lines form triangular cells; the
[illegible]



[illegible]
[illegible]







B C
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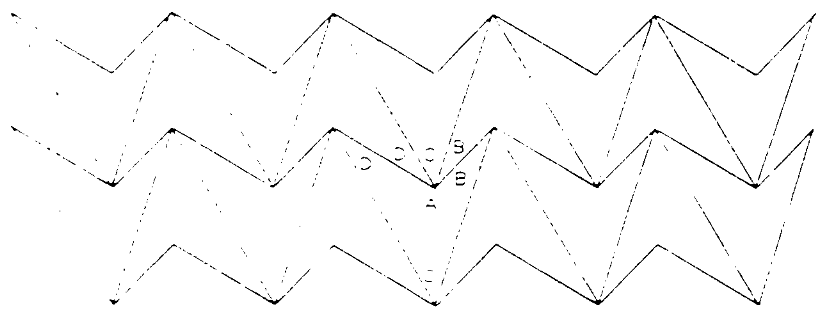
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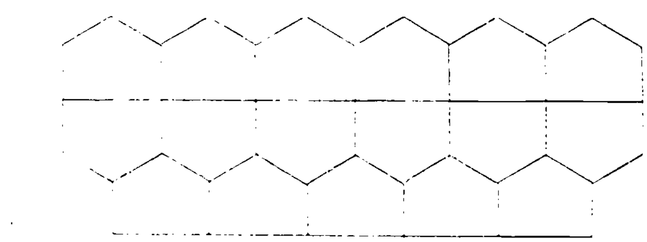
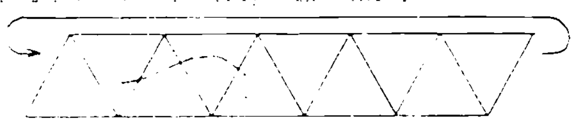
A





ANSWERS

SPHERE-PACKING

1. [illegible] that two [illegible]

2. Twelve. (Four in its own layer, four above and four below.)

3. No. The resulting polyhedron would have to be an icosahedron, which has triangular faces. But the polyhedron with vertices at the contact points has two square faces. The four points of contact with the upper layer, for instance, are vertices of a square face.

4. [illegible].

5. [illegible] (... in its own layer, ... above, ... below.)

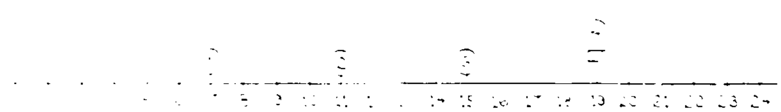
6. [illegible] has square faces, though they are a little harder to see than in the previous case. Alternatively, observe that the polyhedron is symmetric with respect to the plane through the six contact points in the sphere's own layer. Therefore, by symmetry an even number of faces must come together at each of these six points. But the icosahedron has an odd number of faces at each vertex.

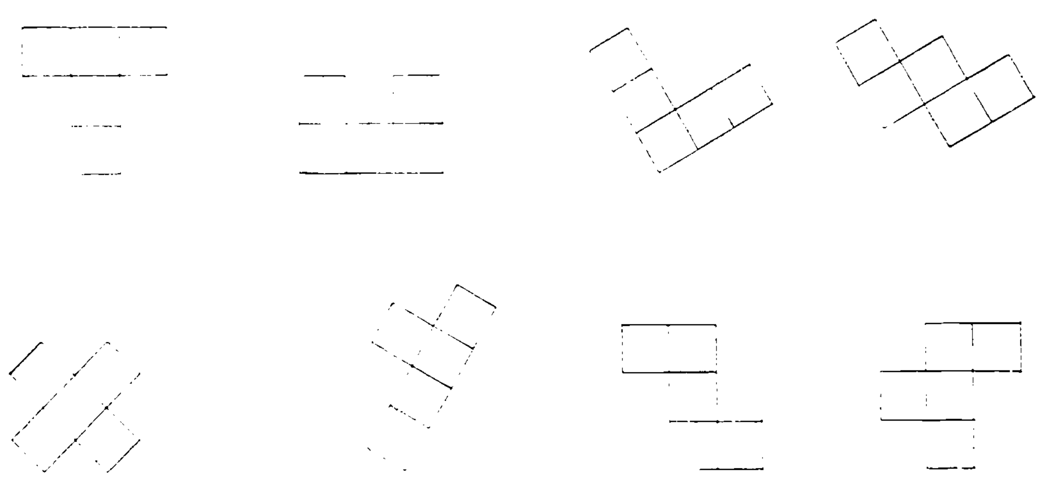
7. Yes.

8. Two.

9. [illegible].

[illegible]





	2	3	5	7	11	13	17	19	23	29
2										
4										
8										
16										
32										
64										
128										
256										
512										
1024										
2048										

This chart provides many opportunities for students to recognize patterns and make discoveries which are related to rich areas in mathematics.

A teacher might start out by listing the powers of 2 down the chalk board in order and ask for students to continue giving entries according to the pattern the teacher has in mind. A similar activity can be performed for the primes across the top.

At this point we will diverge briefly from the major topic at hand to record an activity, brought out in our discussions, which relates to prime numbers.

The children are given some type of counters--pennies, pebbles, checkers, chips, pegs, etc.--and asked to see how many different rectangular arrays they can form with a given number of counters. Any n by m array is considered to be the same as any other n by m array as well as the same as any m by n array. Each child should conduct his own experiments and summarize his findings in a table like this one:

Number of Tokens	1	2	3	4	5		7	8	9	10	11 ...
Number of Rectangular Arrays	1	.	1	2	1	2	1	2	2	2	1 ...

Students may now be told that those numbers for which there is only one rectangular array are the prime numbers (except for the number one which is or isn't depending upon the definition or particular author). This activity could be carried on along with getting kids to discover the pattern across the top of the chart. Upon looking at the table generated from investigating rectangular arrays, a student might be asked questions similar to the following.

Is there a number for which more than two rectangular arrays may be formed?

How many rectangles can be formed with six tokens? How many factors has six?

How many rectangles can be formed with nine tokens? How many factors has nine?

Find a number between 50 and 100 which has exactly 3 factors. How many such numbers are there?

The activity of forming rectangles can also be lowered down into the primary grades to teach multiplication. cf. The Advanced Montessori Method Vol. II by Maria Montessori.

In filling the powers of two-prime chart, column by column students recogniz many patterns and are rewarded by this recognition since following the patterns greatly speeds their task. However, soon even the patterns become difficult to construct. This motivates students to make further discoveries which are immediately rewarding.

A very satisfying discovery is to abandon the set rules to divide a prime into the various powers of 2 and merely begin to double the previous remainder and record the excess above any multiple of the prime in question. This serves as a good introduction or application to modular arithmetic as well as some of the basic theorems on congruence such as Fermat's Little Theorem:

When p is prime and a is not divisible by

$$a^{p-1} \equiv 1 \pmod{p}$$

H. Wills

A PROBLEM IN DIVIDING FRACTIONS

Take the first five prime numbers -- 2, 3, 5, 7, 11 -- and write them with four division signs of unequal length in the following way:

$$\cfrac{2}{\cfrac{\frac{3}{5}}{\frac{7}{11}}}$$

The value of this expression is given by the rules that (a) the longest division sign is applied first, and (b) this rule is repeated until a simple fraction is obtained. Successively, one gets

$$2 \div \frac{\frac{3}{5}}{\frac{7}{11}} = 2 \times \frac{7}{11} \div \frac{3}{5} = \frac{2 \times 5 \times 7}{3 \times 11}$$

The final result need not be multiplied out.

By putting the division signs in a different sequence, a different value may be obtained. Thus,

$$\cfrac{2}{\cfrac{3}{\cfrac{5}{\frac{7}{11}}}} = 2 \div \cfrac{3}{\cfrac{5}{\frac{7}{11}}} = 2 \times \cfrac{5}{\frac{7}{11}} \div 3$$

$$= 2 \times 5 \div \frac{7}{11} \div 3 = \frac{2 \times 5 \times 11}{3 \times 7} .$$

How many different arrangements of signs are possible? Do they all give different answers?

Try

$$\cfrac{2}{\cfrac{\frac{3}{5}}{\frac{7}{11}}}$$

How many distinct <u>values</u> are possible?

The primes are used so that all repetitions occur only as a result of equivalent arrangements of signs. The problem provides examples of non-associativity in arithmetic operations, gives practice in handling division by multiplying reciprocals, tests accuracy in handling compound fractions, and is challenging from the combinatory point of view. For this latter aspect, have the students look at the same problem with 2, 3, or 4 division signs and try to find the rule for the number of distinct values obtained in the general case.

W. Jacobs

LOGIC PUZZLES

"I don't like spinach and I'm glad I don't
because if I did, I'd eat it and I hate it."

Logic puzzles can be useful in overcoming the purely linguistic difficulty of expressing things accurately in the ambiguous, redundant, variable medium of English. The syllogism puzzles of Lewis Carroll are especially good for this because the absurdity of their content serves to emphasize that only the form is of any importance in logic. These puzzles can be found, arranged in a graded sequence, in "Logical Nonsense", Putnam, 1934, pp. 505-546. For use with modern children, they need to be sifted and modified somewhat. In particular it might be a good idea to start with equivalent forms of single statements (which Lewis Carroll doesn't have) before going on to multi-statement puzzles. Following are a few examples (variants of L. C. assertions). For teaching purposes one would want many dozens of these to be worked first by common sense then by more systematic methods.

A. Equivalent forms of single statements. Do the two statements say the same thing or different things? (Answers below)

1. No large birds live on honey. If a bird lives on honey, it is not large.
2. No one, who forgets a promise, fails to do mischief. Anyone who forgets a promise does mischief.
3. All, who are anxious to learn, work hard. All, who work hard, are anxious to learn.
4. Prudent travellers carry plenty of small change.
   Travellers without small change are imprudent.
5. No child is healthy who takes no exercise.
   Children who exercise are healthy.
6. Some elderly ladies are talkative
   Some talkative persons are elderly ladies.
7. Nobody, who really appreciates Beethoven, fails to keep silent when the Moonlight Sonata is being played.
   People, who keep silent when the Moonlight Sonata is being played, really appreciate Beethoven.

8. No drug is useful in a toothache, unless it relieves pain.
A drug that relieves pain is useful in a toothache.

9. None but a hop-scotch player knows real happiness.
All who know real happiness are hop-scotch players.

10. Whenever I do not take my umbrella it rains.
Whenever it does not rain, I take my umbrella.

Answers:

1.	same	6.	same
2.	same	7.	different
3.	different	8.	different
4.	same	9.	same
5.	different	10.	same

After some practice with these, one can go on to puzzles involving two or more statements such as these:

B. Decide whether the conclusion follows from the two statements. (Answer below)

11. "I saw it in a newspaper."
"All newspapers lie."

Conclusion: It was a lie.

12. Every eagle can fly.
Some pigs cannot fly.

Conclusion: Some pigs are not eagles.

13. All, who are anxious to learn, work hard.
Some of these boys work hard.

Conclusion: Some of these boys are anxious to learn.

C. In each of the following examples, draw a conclusion, if possible.

14. All ducks waddle
Nothing that waddles is graceful.

15. Some unkind remarks are annoying.
No critical remarks are kind.

16. Canaries, that do not sing loud, are unhappy.
No well-fed canary fails to sing loud.

17. All puddings are nice.
    This dish is a pudding.
    No nice things are wholesome.

18. Nobody who really appreciates Beethoven fails to keep silent while the Moonlight Sonata is being played.
    Guinea-pigs are hopelessly ignorant of music.
    No one who is hopelessly ignorant of music ever keeps silent while the Moonlight Sonata is being played.

Answers:

11. Conclusion does not follow.
    ("All newspapers lie" is taken to mean "All newspapers sometimes lie." It could conceivably mean "All newspapers always lie." Such ambiguities should be avoided at first, later discussed.)
12. Conclusion follows.
13. Conclusion does not follow.
14. All ducks are ungraceful.
15. No conclusion.
16. Some ill-fed canaries are unhappy. (Lewis Carroll does not admit the possibility that the set of soft-singing canaries is empty. By modern standards one should admit this possibility. The important thing, though, is to be clear about the rules of the game. One implicit rule is that "some" means "one or more." Another implicit rule is that everything has at most two values: Canaries are either well-fed or ill-fed. Indifferently fed canaries do not exist.)
17. This dish is not wholesome.
18. Guinea-pigs do not really appreciate Beethoven.

These are a few examples to illustrate the tone and difficulty of these puzzles. For teaching purposes one should have dozens of them. After doing a great many by common sense one could introduce the set interpretation together with the usual visual aids (Venn diagrams). This should give the pupils quite a sense of power and enable them to go on to harder puzzles involving many statements. Several such are given in Logical Nonsense (loc. cit) including one with twenty statements. Other multi-statement logic puzzles can be found in the standard puzzle books, but without the Carrollean whimsy they often seem stodgy and artificial.

Because of its importance in mathematics, one should also practice a lot with the if .... then form. In particular the children should learn the work-horse rules about converses and negations.

Examples: In each set which statements are equivalent?

1. No child is healthy who takes no exercise.
   If a child takes exercise, then he is healthy.
   If a child is not healthy, then he takes no exercise.

2. No country, that has been explored, is infested with dragons.
   If a country is infested with dragons, then it has not been explored.
   If a country has not been explored then it is infested with dragons.
   If a country has been explored, then it is not infested with dragons.

One can, of course, go on to more formal logic systems, Boolean Algebra, switching circuits, etc., but this is really a different subject, mathematics rather than linguistics. Puzzles of the Lewis Carroll kind are primarily linguistic: they concern equivalences between forms of English sentences. Skill in handling these equivalences is essential if one is to talk about mathematics in the English language.

Following is another logic puzzle that I happen to know and like.

The District Attorney

This is basically a puzzle, but it can be made into a guessing game by having someone in the know act out the part of the D.A. Even when people see the D.A. in action they often have difficulty divining his strategy, which can be made more mysterious by using different but logically equivalent questions each time through.

One of three suspects is guilty. The innocent ones can be counted on to tell the truth, but the guilty one may or may not tell the truth. The D.A. is to find out who is guilty with just two yes-no questions. At first glance it may seem impossible to get any information at all, but the D.A. does it, e.g., as follows:

D.A.: (to suspect A) All right A, either you or B did it, right?

A: No

D.A.: Then I can only conclude that C did it. (to suspect B): Will you confirm that C did it?

B: Yes

D.A.: Of course. C did it.

Alternative scenario:

D.A.: (to C) You look honest. Now tell me the truth. Did A do it?

C: No.

D.A.: (to A) If C were guilty he would have said, "yes" in order to frame you. Therefore it must be B. B did it, right?

A: No

D.A.: All right C. The jig is up. Honesty will get you nowhere.

C: But what if I had lied?

D.A.: I would have found out just the same. Dishonesty is no better than honesty.

C: You can't win.

D.A.: True.

False Proof by Induction

At the high school level after students have had experience with mathematical induction they should try this one:

Theorem: In any set of marbles all the marbles are the same color.

Proof (by induction) Let n be the number of marbles in the set. The theorem is certainly true if $n = 1$.

Induction: Suppose the theorem is true for n. Then it is surely true for $n + 1$, for if I remove any marble from a set of $n + 1$, the remaining ones, constituting a set of n, must all be the same color. Since this is true no matter which marble I remove, all $n + 1$ marbles must be the same color.

F. W. Sinden

DOES THE ORDER MAKE A DIFFERENCE?

1. In a single purchase, you are offered 3 successive discounts of 20 percent, 10 percent, and 5 percent, and can take them in any order that you wish. What order would you choose?

2. Arthur and Bob start a game with equal amounts of money. Arthur loses the first game and pays Bob 20 percent of his money. Then Bob loses the second game and pays Arthur 20 percent of the amount Bob has. Do they again have equal amounts of money?

3. Mr. Jones has two houses, which cost him the same amount. He sells one house at a 10 percent profit to Mr. Allen, who resells it to Mr. Baker at a 10 percent loss. Mr. Jones sells his second house at a 10 percent profit to Mr. Dahl. Which paid more, Mr. Baker or Mr. Dahl?

IDENTIFICATION PROBLEMS

The class of tricks based on what is generally called Gergone's Pile Problem provide an example of the power of numerical coding and an application of place system ideas. A typical version is the following: The "magician", A, asks B to deal 27 distinguishable cards in 3 equal piles, to select a card mentally and to announce which pile it is in. A then tells B to reassemble the cards by placing one pile on top of the other in any order. A notes the order in which the piles are assembled. After two repetitions of this cycle A tells B the position of his card in the reassembled deck.

If this trick is introduced to a fourth or fifth grade class in its 2 pile, 4 card form an analysis by the class should be possible and can be made to relate to binary numeration. The binary form and the general case can be investigated at an appropriate later time.

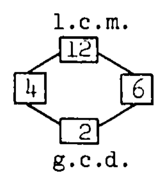
A complete discussion of several variations of this trick can be found in Martin Gardner's Mathematics, Magic, and Mystery (Dover, 1956).

W. Jacobs

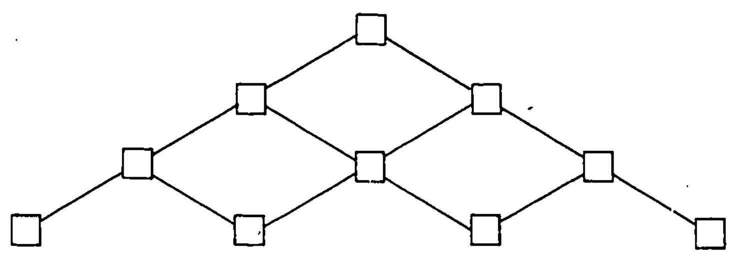
A GAME WITH FACTORS AND MULTIPLES
(Grades 4-6)

The type of game outlined below depends solely on the multiplicative aspect of the positive integers. The objective is to explore factor relationships and incidentally to reinforce direct recall of the multiplication facts basic to computation.

The game is based on the g.c.d. and l.c.m. diagram exemplified by



To make a game consider a triangular array of say 10 boxes.



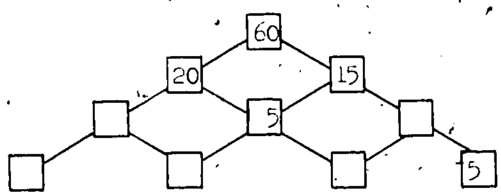
The game is played by 2, or perhaps 3, players, one of whom might be the teacher. Player A begins by filling in the top box. The other "players" in turn fill in the other boxes. In a simple version the only rule governing permissible entries is that the number put in the top box may never be used again.

The Play

After an entry is made any other player may challenge the player making it to show that the configuration is part of a "factor diagram". If the latter succeeds he wins (or scores so many points). If he fails he loses (or his challenger scores so many points). At any time a player believes his play makes the configuration part of a unique factor diagram he wins (or scores) upon convincing the other players of his assertion.

Here is a play.

(4)
(3)
(2)
(1)



A	60(4)		20(3)		15(3)
B		5(1)		5(2)	

This last play can be challenged.

Additional rules for admissible entries may be added. It may be required: for example, that

i) The entries in row (1) must be relatively prime in pairs, or

ii) The entries in row (1) must be primes or 1, or

iii) The number 1 may not be used.

W. Lister

204

A STOCHASTIC NUMBER LINE GAME

There are many commercial games which are essentially stochastic number line games. Typically, in these games the players move playing pieces along a path on the playing board, the length of the move being determined by a throw of a die or a pair of dice. The interest in such games is usually supplied by [illegible] rewards and penalties for stopping at specific position along the path. While these games may interest children - and sometimes adults - the games are not particularly instructive since the play is purely a chance phenomenon [illegible] understanding of the probabilistic principles underlying the [illegible] way aid the perceptive player. In order to make the game instructive, [illegible] with respect to probability, it should be possible for the perceptive [illegible] principles of probability to improve the [illegible]. We [illegible] example of such a game.

[illegible] the number line although any discrete [illegible] will [illegible]. Any number of players may play and [illegible] players in turn throw a pair of dice [illegible] equal to the number of dots [illegible] of being able to choose either a [illegible] that enables a player to improve [illegible] game is to reach the points numbered [illegible] his turn will throw the dice, [illegible] then will proceed to [illegible], [illegible] this does not count and he must [illegible]. The first player to reach a given [illegible].

[illegible] knowledge about the distribu- [illegible] are thrown to improve his [illegible] it is clearly advantageous [illegible] since he will have a much better [illegible] will of throwing a [illegible].

[illegible] the best strategy is much simpler. [illegible] in the direction of the position [illegible] to analyze the best strategy [illegible] and four [illegible] its faces, or if it has one 1, two 2's, and three 3's.

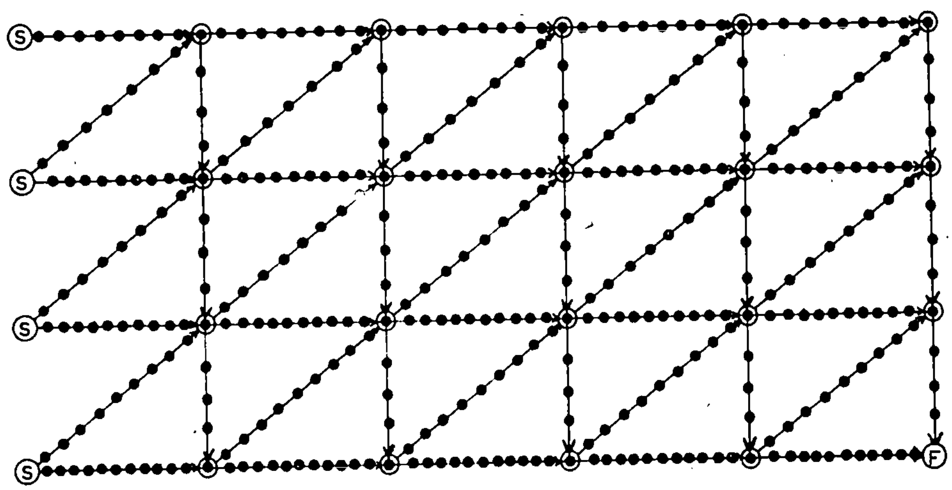
It is clear that games of this type give students considerable incentive to learn and use probability principles which will improve their chances of winning. With suitable choices of the basic distribution much of basic probability can be learned in this way. Furthermore, the kind of probabilistic reasoning required is very similar to the kind needed to make effective probability judgments in ordinary life situations.

R. P. Dilworth

GAMES ON DIRECTED GRAPHS

Stochastic games on the number line are a special case of stochastic games on directed graphs. Again there exist a number of commercial games which have the players move through a network, each move being determined by the throw of a die or a pair of dice. These games are likewise not particularly instructive since the play is usually a purely chance affair. However, games on directed graphs can easily be constructed in which an understanding of the underlying probability can be used to improve the strategy of play. Such games are potentially instructive particularly with regard to basic principles of probability. We give an example of a comparatively simple stochastic game on directed graph.

The playing board has a diagram as follows:



The game is played by four players who start in the four circular positions label S. They may draw lots to determine who starts in each position. They play in the order I, II, III, IV and, in his turn, each player throws a pair of dice to determine how far he will move along the directed graph. He may move along any path consistent with the arrows. His objective is to reach the finish position F in the fewest number of moves. The player first passing through a junction secures it for himself and other players may not follow paths passing through that junction.

The first player reaching F makes a score of 60 - n where n is the number of dots he has traversed in reaching the finish. All other players score zero. Note that the first player to reach F may still get a negative score.

This game clearly has a great many more strategic possibilities than the previous number line game. For example, player IV gains nothing by heading to the finish position directly since his score is then zero, but by taking a path higher into the graph he increases his chances of being cut off. Likewise, player I must decide whether he should take a path around the outside where he can make a good score but runs the risk of being cut off or taking a path into the graph and doing some cutting off himself. Perhaps the numbers of dots between junctions will have to be altered if a wide variety of strategies is to be obtained. Certainly, the distribution of the possible steps in each play must be used in estimating the effectiveness of a given strategy.

R. P. Dilworth

ELEMENTARY THEORY OF AREAS

There is a good deal of puzzle value in dissection problems and such experience should certainly help develop spatial geometric intuition especially the notions of Euclidian motions or congruences.

A pair of polygonal regions A and B are "equivalent" if A can be dissected into polygonal regions which can in turn be reassembled to form B. For example, in FIG. 1, A is equivalent to B, B to C and A to C.

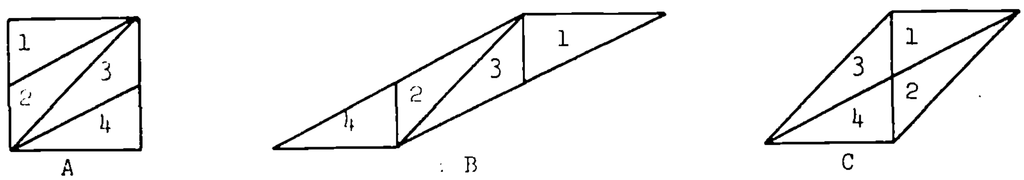


FIGURE 1

It is clear that any pair of equivalent polygonal regions have the same area; but the elegant thing is that the converse is also true: thus: "Every pair of polygonal regions with the same area are equivalent." It is this theorem which is illustrated here.

1. Show that the two parallelogram A and B, (see FIG. 2), which have the same heights and bases, are equivalent.

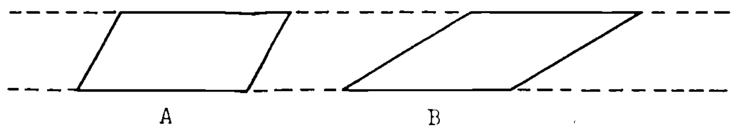
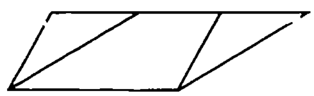
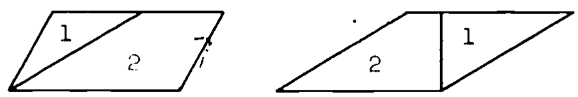


FIGURE 2

(Answer) superimpose bases

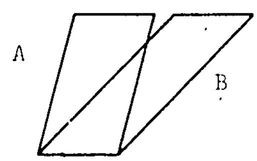


and a proper dissection becomes clearer:

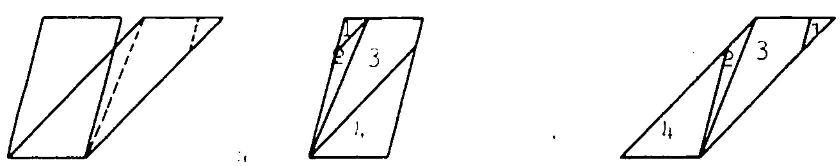


2. Show that the two parallelograms A and B in FIG. 3 (in which they are illustrated with their bases superimposed) are equivalent.

FIGURE 3



(Answer) This is similar to problem 1 but requires an extra step. The The following illustration gives the idea:



3. Show that the triangular region A is equivalent to the rectangular region B. (B has half the height of A).

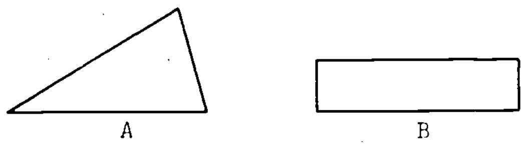
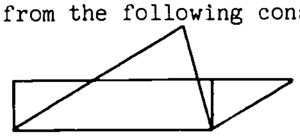


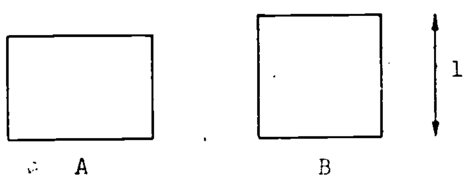
FIGURE 4

(Answer): from the following construction



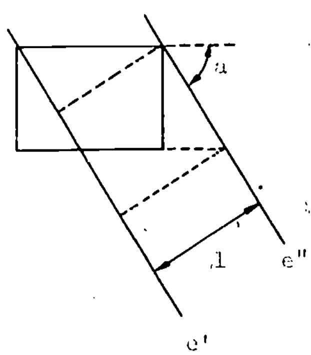
one obtains the dissection :

4. Show that the rectangles A and B in FIG. 5 bound equivalent regions. The height of A is less than one and its length is greater than one. The height of B is equal to one and (since the areas bounded by A and B are equal) the length of B must be the common area of A and B.



FIGURES

(Answer): make use of following construction



choose angle a so that distance between e' and e" is one.

5. Using the devices indicated in problems 1, 2, 3, 4 one can show that any triangle is equivalent to a rectangle of height one. It seems reasonable (and it can in fact be proved) that any polygon can be dissected into triangles; for example see FIG. 6.

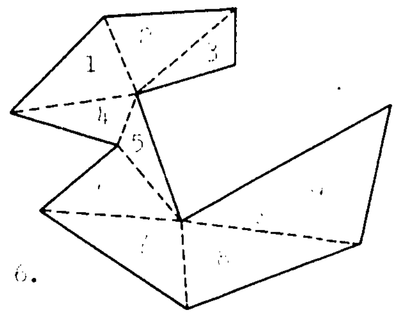
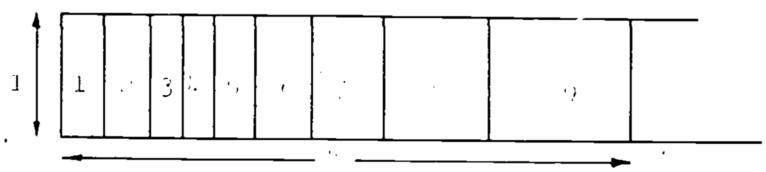


FIGURE 6.

Using the result on triangles one can dissect each of the triangles and rearrange into a rectangle of height one; thus



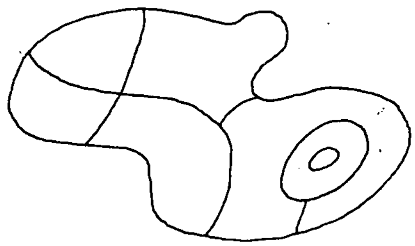
This not only computes the area "a" of the original polygonal region but also goes a long way toward showing the general result mentioned on page 1.

C. J. Titus

MAP COLORING PROBLEMS

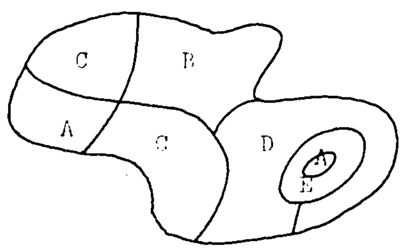
In the following we are concerned with "maps"; for example,

FIGURE 1;



and we are also concerned with the coloring of maps in a special way. We will say a map is "correctly colored" if no pair of bordering countries have the same color; for example, the following is a correct coloring" of the map in FIGURE 1:

FIGURE 2;

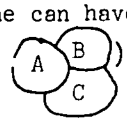


1. Can you "correctly color" the map in FIGURE 1 with fewer than the five colors used in FIGURE 2?

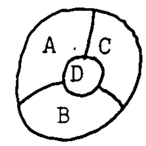
2. What is the least number of colors with which on can correctly color the map in FIGURE 1? (ans: 3)

3. Draw a map which can be correctly colored with two colors. (sample answer: )

4. Draw a map that can be correctly colored with 3 colors but that cannot be correctly colored with 2 colors. (sample answer: the map in FIGURE 2).

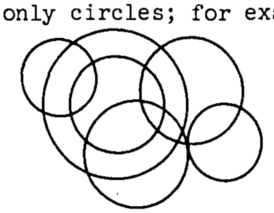
5. How many countries did you have in the map in problem 4? Can you achieve the same result with fewer countries? What is the smallest number of countries one can have in a map and still have the same result? (answer: 3; sample map 

)

6. Draw a map that can be correctly colored with 4 colors but that cannot be correctly colored with 3 colors. Can you achieve the same result with fewer countries? What is the smallest number of countries one can have in a map and still have the same result? (answer: 4; sample map 

)

7. Make a map using only circles; for example,



Correctly color your map with two colors. Notice that you cannot find such a "circle map" that requires more than 2 colors. Can you see why it is that every circle map can be colored with two colors? (sample answer: write a number in each country which is the number of circles in which that country is contained. Color the even numbered countries one color and the odd numbered countries the other color.

C. J. Titus

A COMBINATORIAL PROBLEM SOLVED BY GEOMETRY

Consider the eight (2^3) triplets of 0's and 1's:

000
001
010
011
100
101
110
111

Can you arrange these in a cycle so that neighboring triplets differ in only one place? This problem arises in computer design where the 0's and 1's represent states of on-off devices (e.q. switches, lights, cores). A triplet of such devices is supposed to run through all of its eight states over and over again. For engineering reasons it may be awkward to switch two or more of the devices simultaneously. Therefore one would like to find a sequence that requires switching only one at each step.

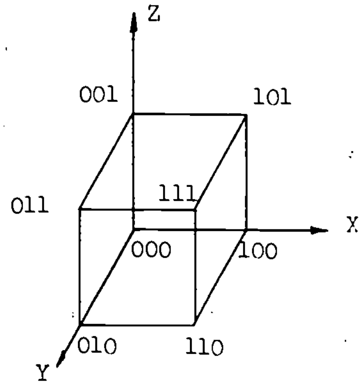
This looks like a hard problem. The number of possible cycles is enormous.

$$\frac{8!}{8} = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7 = 5040$$

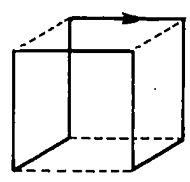
(Division by 8 because cyclic permutations indistinguishable)
It turns out, though, that a geometric interpretation makes the problem easy. Consider the triplets of 0's and 1's to be coordinates of points in 3 space.

215

The eight points represented by the triplets lie at the vertices of a cube.



Neighboring vertices (those joined by an edge) differ in exactly one coordinate. The problem then, is to find a closed path along the edges of the cube which passes through each vertex once. Such a path (Hamilton line) is easy to find:



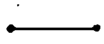
Tracing out this path, one gets the solution:

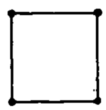
000
001
101
100
110
111
011
010

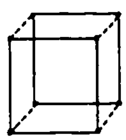
All other Hamilton lines are rotations of the one shown above.

This problem offers an excellent opportunity to discuss higher dimensional geometry. To solve the problem with quadruples of 0's and 1's one considers Hamilton lines on a 4 - cube. To show what a 4 - cube is, one can display cubes of 0, 1, 2 and 3 dimensions, note the induction principle, use it to get the 4 - cube.

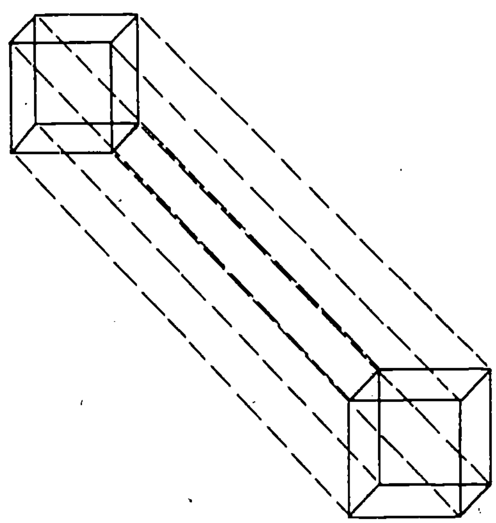
0 - cube

1 - cube  (2 0-cubes joined by a segment)

2 - cube  (2 1-cubes with corresponding vertices joined)

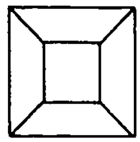
3 - cube  (2 2-cubes with corresponding vertices joined)

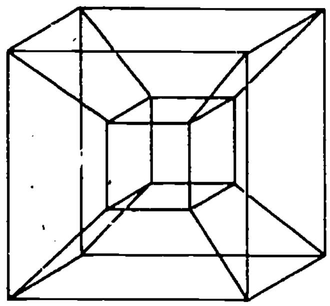
4 - cube (2 3-cubes with corresponding vertices joined)



217

The drawing is neater if you put one cube inside the other:

3 - cube:  (one 2-cube inside the other)

4 - cube:  (one 3 - cube inside the other)

The construction of the Hamilton line can be generalized. Observe that the Hamilton line on a 3-cube first traverses one of the 2-cubes, then jumps to the other 2-cube, traverses it, jumps back to starting point. On the 4-cube: First traverse a 3-cube, jump to the other 3-cube, traverse it, jump back to starting point.

F. W. Sinden